Section 1.7 Bifurcations

Many mathematical models include differential equations with parameters along with other variables.

We want to consider such equations and investigate how solutions change (depend on) the parameter(s) as they vary.

We limit the discussion to *autonomous* equations with one parameter.
Given the simple dynamics we've seen from vector fields on the real line, it seems there isn't much to one-dimensional systems.

These systems become much more interesting if solutions are made to depend on parameters.
The qualitative structure of solutions can change as parameters are varied.

These qualitative changes in the dynamics of systems are called *bifurcations*,

The parameter values at which these changes occur are called *bifurcation values*. 
Let's look at a fundamental kind of bifurcation. It is often called a *saddle-node bifurcation*.

In this case, as a parameter is varied, two equilibria move toward each other, collide, and mutually annihilate. Consider the first-order equation

\[
\frac{dx}{dt} = \alpha + x^2, \quad \alpha \in \mathbb{R}.
\]
First we find the equilibria:

Set \( \frac{dx}{dt} = \alpha + x^2 = 0 \Rightarrow x^2 = -\alpha \Rightarrow x = \pm \sqrt{-\alpha} \)

Then consider various values of \( \alpha \):

- If \( \alpha > 0 \) there are no equilibria.
- If \( \alpha = 0 \) there is one equilibrium solution, \( x = 0 \).
- If \( \alpha < 0 \) there are two distinct equilibrium solutions,
  \[ x = +\sqrt{-\alpha} \text{ and } x = -\sqrt{-\alpha}. \]
Linearization theorem

Suppose $y_0$ is an equilibrium point of the differential equation $\frac{dy}{dt} = f(y)$ with $f$ continuously differentiable.

Then,

- if $f''(y_0) < 0$, then $y_0$ is a sink (stable);
- if $f''(y_0) > 0$, then $y_0$ is a source (unstable); or
- if $f''(y_0) = 0$, need more information to determine the type of $y_0$. 

Two equilibria 
\( \alpha < 0 \)

\[ \frac{dx}{dt} = \alpha + x^2 \]
To classify the equilibria, let $\frac{dx}{dt} = f_\alpha(x) = \alpha + x^2$.

Then

$$f'_\alpha(x) = 2x,$$

and by our linearization theorem, if $x_0$ is an equilibrium point and $x_0 = -\sqrt{-\alpha} < 0$,

then $f'_\alpha(x) < 0 \Rightarrow x_0$ is a sink.

and if $x_0 = \sqrt{-\alpha} > 0$,

then $f'_\alpha(x) > 0 \Rightarrow x_0$ is a source.
As $\alpha$ approaches 0 from below, the parabola moves up and the two equilibria move toward each other.

When $\alpha = 0$,

the equilibria collapse into a half-stable equilibrium point at $x = 0$. 
This type of equilibrium is very fragile - it vanishes as soon as $\alpha > 0$, and then there are no equilibria at all.

We say a bifurcation occurred at $\alpha = 0$.

The vector fields for $\alpha > 0$ and $\alpha < 0$ are qualitatively different.

Thus $\alpha = 0$ is a bifurcation value.
Two equilibria
$\alpha = -2$
One equilibrium

$\alpha = 0$

\[
\frac{dx}{dt} = \alpha + x^2
\]

\[
\frac{dx}{dt} \geq 0 \quad \forall x
\]
One equilibrium
\[ \alpha = 0 \]
No equilibria
\(\alpha > 0\)

\[
\frac{dx}{dt} = \alpha + x^2
\]

\[
\frac{dx}{dt} > 0 \quad \forall x
\]
No equilibria, \( \alpha = 2 \)

\[
\frac{dy}{dt} = 2 + y \cdot y
\]
The most common representation is to invert the axes so that the independent variable, $\alpha$, is plotted horizontally. This is called a *bifurcation diagram*. 
Example problem:

Draw the bifurcation diagram for the equation:

\[ \frac{dy}{dt} = y(1 - y) - \alpha, \quad \alpha \text{ a real parameter.} \]

First find the equilibria ...
Set \( \frac{dy}{dt} = y(1-y) - \alpha = 0 \) or \( y^2 - y + \alpha = 0 \).

By the quadratic formula \( y = \frac{1 \pm \sqrt{1 - 4\alpha}}{2} \).

If \( \alpha < 1/4 \) there are two equilibria, \( y = \frac{1 + \sqrt{1 - 4\alpha}}{2} \) and \( y = \frac{1 - \sqrt{1 - 4\alpha}}{2} \).

If \( \alpha = 1/4 \) there is one equilibrium point, \( y = \frac{1}{2} \).

If \( \alpha > 1/4 \) there are no equilibria.
\[ \alpha = 0 \]
$\alpha = 0.30$
\[ \alpha = 0.15 \]
$\alpha = 0.19$
\[ \alpha = 0.25 \]

\[ \frac{dy}{dt} = y(1 - y) - a \]
There was a large qualitative change in the solutions as the parameter $\alpha$ passed left to right through the critical value $\alpha = 0.25$.

Note that when there were two equilibrium solutions all trajectories either side of the larger one moved *toward* the equilibrium solution as $t \to \infty$. In this case, the equilibrium solution is called a *sink*.

In contrast, all trajectories either side of the smaller one moved *away* from the equilibrium solution as $t \to \infty$. In this case, the equilibrium solution is called a *source*. 
Summary of equilibria analysis for \( \frac{dy}{dt} = y(1-y) - \alpha \):

Equilibrium points occur when \( \frac{dy}{dt} = y(1-y) - \alpha = 0 \Rightarrow y = \frac{1\pm\sqrt{1-4\alpha}}{2} \).

If \( 1-4\alpha < 0 \Rightarrow \alpha > \frac{1}{4} \) and there are no equilibria.

If \( 1-4\alpha = 0 \Rightarrow \alpha = \frac{1}{4} \) and there is one equilibrium point, \( y = \frac{1}{2} \).

If \( 1-4\alpha > 0 \Rightarrow \alpha < \frac{1}{4} \) and there are two equilibria,

\[ y_1 = \frac{1+\sqrt{1-4\alpha}}{2} \quad \text{and} \quad y_2 = \frac{1-\sqrt{1-4\alpha}}{2} \]
Now we need to complete the bifurcation diagram which lies in the $\alpha$-$y$ plane.

The bifurcation diagram exhibits phase lines near the bifurcation value and demonstrates the changes that the phase line goes through as the parameter $\alpha$ passes through the bifurcation value.
For our example, the bifurcation value is $\alpha = \frac{1}{4}$. If $\alpha < \frac{1}{4}$, there are two real equilibria, if $\alpha = \frac{1}{4}$ there is one equilibrium point, and if $\alpha > \frac{1}{4}$, there are no equilibrium points.

The qualitative nature of the phase lines changes when $\alpha = \frac{1}{4}$, making it the *bifurcation value*. 
\[
\frac{1 + \sqrt{1 - 4\alpha}}{2}
\]

\[
\frac{1 - \sqrt{1 - 4\alpha}}{2}
\]

\[\alpha = 1/4\]
End Presentation