3.5 Special Cases: Repeated and Zero Eigenvalues
Equal Real Eigenvalues

We have the following cases:

1. $\lambda_1 = \lambda_2 > 0$

2. $\lambda_1 = \lambda_2 < 0$

3. $\lambda_1 = \lambda_2 = 0$
Suppose $\lambda_1 = \lambda_2 \neq 0$. If we can find distinct eigenvectors $\vec{X}_1$ and $\vec{X}_2$ which are linearly independent, then the general solution can still be written in the form

$$
\vec{Y}(t) = c_1 e^{\lambda_1 t} \vec{X}_1 + c_2 e^{\lambda_2 t} \vec{X}_2 = c_1 e^{\lambda_1 t} \vec{X}_1 + c_2 e^{\lambda_1 t} \vec{X}_2
$$

$$
= e^{\lambda_1 t} \left( c_1 \vec{X}_1 + c_2 \vec{X}_2 \right).
$$
Example for this case:

\[
\frac{d\vec{Y}}{dt} = A\vec{Y} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \vec{Y}, \quad \vec{Y} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}
\]
Only straight-line solutions are on the x-axis
What about the eigenvalues for this system?

\[ \text{Det}(A - \lambda I) = (\lambda + 2)^2 = 0 \]

So we have a repeated eigenvalue \( \lambda = -2 \).

For the associated eigenvectors we solve \( A\vec{X}_0 = -2\vec{X}_0 \)

for \( \vec{X}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \Rightarrow \begin{cases} -2x_0 + y_0 = -2x_0 \\ -2y_0 = -2y_0 \end{cases} \Rightarrow y_0 = 0 \)

Thus all the eigenvectors corresponding to the eigenvalue \( \lambda = -2 \) lie on the \( x - \)axis.
Since all the eigenvectors corresponding to the eigenvalue $\lambda = -2$ lie on the $x$–axis, all the straight-line solutions also lie on the $x$–axis.

We select a representative eigenvector ...

$$\tilde{X} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} = x_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$ Setting $x_0 = 1$ gives $\tilde{X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and so the solution corresponding to this eigenvalue, eigenvector pair is

$$Y_1(t) = e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ ... but we need another to complete our general solution!
Now if we try to write our solution in the form

\[ \bar{Y}(t) = c_1 e^{\lambda t} \bar{X} + c_2 e^{\lambda t} \bar{X}, \]

we would get

\[ \bar{Y}(t) = c_1 e^{\lambda t} \bar{X} + c_2 e^{\lambda t} \bar{X} = (c_1 + c_2) e^{\lambda t} \bar{X} = ke^{\lambda t} \bar{X} \]

This can not possibly be the general solution for a 2-D system since we have only one arbitrary constant!
We must find another solution to the system that is \textit{independent} of the one solution we have found.

An independent solution means one which is not a scalar multiple of the solution we already have.

If we do find another eigenvector satisfying the independence requirement, then the solution can still be written in the form \( \vec{Y}(t) = c_1 e^{\lambda t} \vec{X}_1 + c_2 e^{\lambda t} \vec{X}_2 \).
All solution curves approach the sink at the origin though there is only one line of eigenvectors
An approach to find a second linearly independent solution is suggested by the strategy used in the second order case with a repeated eigenvalue. Recall that we produced a second independent solution by multiplying the initial solution by $t$.

If we try a solution of the form $te^{\lambda t}\vec{V}$ - it will not work. So we try instead that solution plus $e^{\lambda t}\vec{W}$ where $\vec{W}$ is a nonzero vector. Then $\vec{Y}(t) = te^{\lambda t}\vec{V} + e^{\lambda t}\vec{W}$ is assumed to be a solution.
Back to our example system

\[ \frac{d\vec{Y}}{dt} = A\vec{Y} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \vec{Y} \]
or in component form

\[ \frac{dx}{dt} = -2x + y, \quad \frac{dy}{dt} = -2y. \]

In this case we can solve the second differential equation directly ...

\[ \frac{dy}{dt} = -2y \Rightarrow y(t) = y_0 e^{-2t} \]

where \( y_0 \) represents the initial value of \( y(t) \).

Now we can substitute \( y(t) = y_0 e^{-2t} \) into

\[ \frac{dx}{dt} = -2x + y \Rightarrow \frac{dx}{dt} = -2x + y_0 e^{-2t} \Rightarrow x(t) = y_0 te^{-2t} + x_0 e^{-2t} \]

where \( x_0 \) is the initial value of \( x(t) \).
And we can write the general solution in vector form as

\[
\vec{Y}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} y_0te^{-2t} + x_0e^{-2t} \\ y_0e^{-2t} \end{bmatrix}
\]

This solution can also be written as

\[
\vec{Y}(t) = e^{-2t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + te^{-2t} \begin{bmatrix} y_0 \\ 0 \end{bmatrix}
\]

where \( y_0 \) represents the initial value of \( y(t) \) and \( x_0 \) is the initial value of \( x(t) \).
Form of the general solution for

\[ \frac{d\vec{Y}}{dt} = A\vec{Y} \] where matrix \( A \) has a repeated eigenvalue \( \lambda \).

We try a solution of the form \( \vec{Y}(t) = te^{\lambda t}\vec{V}_0 + e^{\lambda t}\vec{V}_1 \)

where \( \vec{Y}(0) = \vec{V}_0 \) is the initial condition of \( \vec{Y}(t) \).

Let's plug our solution guess into \( \frac{d\vec{Y}}{dt} = A\vec{Y} \)

On the left side we have \( \frac{d\vec{Y}}{dt} = \lambda e^{\lambda t}\vec{V}_0 + e^{\lambda t}\vec{V}_1 + t\lambda e^{\lambda t}\vec{V}_1 \)
and
\[ \lambda e^{\lambda t} \vec{V}_0 + e^{\lambda t} \vec{V}_1 + t \lambda e^{\lambda t} \vec{V}_1 = e^{\lambda t} (\lambda \vec{V}_0 + \vec{V}_1) + te^{\lambda t} (\lambda \vec{V}_1). \]

For the right hand side we have
\[ A \vec{Y} = e^{\lambda t} A \vec{V}_0 + te^{\lambda t} A \vec{V}_1 \]

So \[ \frac{d\vec{Y}}{dt} = A \vec{Y} \] becomes
\[ e^{\lambda t} (\lambda \vec{V}_0 + \vec{V}_1) + te^{\lambda t} (\lambda \vec{V}_1) = e^{\lambda t} A \vec{V}_0 + te^{\lambda t} A \vec{V}_1 \]

Equating like terms gives ...
\[ te^{\lambda t} (\lambda \vec{V}_1) = te^{\lambda t} A \vec{V}_1 \Rightarrow A \vec{V}_1 = \lambda \vec{V}_1 \]  \quad (1)

and

\[ \lambda \vec{V}_0 + \vec{V}_1 = A \vec{V}_0 \]  \quad (2)

Equation (1) says \( \vec{V}_1 \) is an eigenvector unless \( \vec{V}_1 = \vec{0} \)

and (2) says that \( \vec{V}_1 = A \vec{V}_0 - \lambda \vec{V}_0 = (A - \lambda I)\vec{V}_0 \).

So \( \vec{Y}(t) = te^{\lambda t} \vec{V}_1 + e^{\lambda t} \vec{V}_0 \) is a solution \( \Leftrightarrow \vec{V}_1 = (A - \lambda I)\vec{V}_0 \)

and \( \vec{V}_1 \) is an eigenvector, unless \( \vec{V}_1 = \vec{0} \).
THEOREM Suppose \( \frac{d\vec{Y}}{dt} = A\vec{Y} \) is a linear system in which the 2 x 2 matrix \( A \) has a repeated real eigenvalue \( \lambda \) but only one line of eigenvectors. Then the general solution has the form

\[
\vec{Y}(t) = te^{\lambda t} \vec{V}_1 + e^{\lambda t} \vec{V}_0,
\]

where \( \vec{V}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \) is an arbitrary initial condition and \( \vec{V}_1 \) is determined from \( \vec{V}_0 \) by \( \vec{V}_1 = (A - \lambda I) \vec{V}_0 \).

If \( \vec{V}_1 = \vec{0} \) then \( \vec{V}_0 \) is an eigenvector and \( \vec{Y}(t) \) is a straight-line solution. Otherwise, \( \vec{V}_1 \) is an eigenvector.
WARNING Don't make the mistake of thinking that the two separate terms $te^{\lambda t} \vec{V}_1$ and $e^{\lambda t} \vec{V}_0$ are solutions. $\vec{V}_1$ is determined by $\vec{V}_0$. Also, $e^{\lambda t} \vec{V}_0$ by itself can only be a solution if $\vec{V}_0$ is an eigenvector.
Back to our example system once again!

\[
\frac{d\vec{Y}}{dt} = A\vec{Y} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \vec{Y}
\]

In this case we had a repeated eigenvalue \( \lambda = -2 \) and there is only one line of eigenvectors. As in the theorem, we let \( \vec{V}_0 \) be the arbitrary initial condition \( \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \). Then

\[
\vec{V}_1 = (A + 2I)\vec{V}_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} y_0 \\ 0 \end{bmatrix}
\]

and we get the general solution

\[
\vec{Y}(t) = e^{-2t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + te^{-2t} \begin{bmatrix} y_0 \\ 0 \end{bmatrix}
\]
\[ \frac{dx}{dt} = -2x + y \]
\[ \frac{dy}{dt} = -2y \]
\[
\begin{align*}
\frac{dx}{dt} &= -2x + y \\
\frac{dy}{dt} &= -2y
\end{align*}
\]
Another example. Find the general solution for

\[
\frac{d\vec{Y}}{dt} = A\vec{Y} = \begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix}\vec{Y}
\]

For the eigenvalues, the characteristic equation is

\[
(-2 - \lambda)(-4 - \lambda) + 1 = (\lambda + 3)^2 = 0
\]

and the eigenvalue is \(\lambda = -3\).

To find an eigenvector, we solve the simultaneous equations

\[
\begin{align*}
-2x - y &= -3x \\
x - 4y &= -3y
\end{align*}
\]

\(\Rightarrow y = x\) and one eigenvector is \(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\).
Straight line solutions along the vector (1,1)
We have the eigenvalue is $\lambda = -3$ and the eigenvectors $
abla = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ satisfy the equation $y_0 = x_0$.

To find the general solution, start with an arbitrary initial condition $
abla_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

Then $\nabla_1 = \left( \begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \nabla_0 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 - y_0 \\ x_0 - y_0 \end{bmatrix}$

The general solution is

$$\nabla(t) = e^{-3t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + te^{-3t} \begin{bmatrix} x_0 - y_0 \\ x_0 - y_0 \end{bmatrix}$$
Both eigenvalues zero

In this case, $\lambda_1 = \lambda_2 = 0$. If there are two linearly independent eigenvectors $\tilde{X}_1$ and $\tilde{X}_2$ then the general solution is

$$\tilde{Y}(t) = c_1 e^{0 \cdot t} \tilde{X}_1 + c_2 e^{0 \cdot t} \tilde{X}_2 = c_1 \tilde{X}_1 + c_2 \tilde{X}_2,$$

a single vector of constants.

If there is only one linearly independent eigenvector $\tilde{X}$ corresponding to the eigenvalue 0, then we can find our vector $\tilde{W}$ and use the formula

$$\tilde{Y}(t) = c_1 e^{\lambda t} \tilde{X} + c_2 \left[ t e^{\lambda t} \tilde{X} + e^{\lambda t} \tilde{W} \right].$$

For $\lambda = 0$, we get

$$\tilde{Y}(t) = c_1 \tilde{X} + c_2 \left[ t \tilde{X} + \tilde{W} \right] = (c_1 + c_2 t) \tilde{X} + c_2 \tilde{W}$$