3.1 Properties of Linear Systems and the Linearity Principle
Linear systems in general, and matrix notation.

\[
\frac{dx}{dt} = ax + by
\]

We consider systems

\[
\frac{dy}{dt} = cx + dy
\]

where \(a, b, c, d \in \mathbb{R}\).

This is a \(2 \times 2\) linear system with constant coefficients.

The system can be written in matrix form:

\[
\begin{bmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{bmatrix} = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix} x(t) \\
y(t) \end{bmatrix}.
\]
Alternate notation is \[ \frac{d\vec{Y}}{dt} = A\vec{Y} \]

with \( \frac{d\vec{Y}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} \), \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), and \( \vec{Y} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \).
Note on multiplying a matrix times a vector.
There are two ways to view this multiplication:
First, the "standard" approach

\[ A\vec{Y} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a \times(t) + b \times y(t) \\ c \times(t) + d \times y(t) \end{bmatrix} \]

An alternate method is to think of the multiplication as by columns

\[ A\vec{Y} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = x(t) \begin{bmatrix} a \\ c \end{bmatrix} + y(t) \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} a \times(t) + b \times y(t) \\ c \times(t) + d \times y(t) \end{bmatrix} \]
Finding equilibrium points for the linear system \( \frac{d\vec{Y}}{dt} = A\vec{Y} \).

\( \vec{Y}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \) is an equilibrium point for \( \frac{d\vec{Y}}{dt} = A\vec{Y} \) if and only if the vector field at \( \vec{Y}_0 \) is the zero vector.

This will be the case if

\[
A\vec{Y}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \Rightarrow \begin{bmatrix} a x_0 + b y_0 = 0 \\ c x_0 + d y_0 = 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
Thus the origin is an equilibrium point for the linear system \( \frac{d\vec{Y}}{dt} = A\vec{Y} \).

The constant function \( \vec{Y}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) \( \forall t \), is a solution to \( \frac{d\vec{Y}}{dt} = A\vec{Y} \).

This solution, called the *trivial solution*, is an equilibrium solution for *every* linear system.
Linearity Principle

Suppose \( \frac{d\vec{Y}}{dt} = A\vec{Y} \) is a system of 1st-order linear differential equations.

1. If \( \vec{Y}(t) \) is a solution of this system and \( k \) is any constant, then \( k\vec{Y}(t) \) is also a solution.

2. If \( \vec{Y}_1(t) \) and \( \vec{Y}_2(t) \) are two solutions of this system, then \( \vec{Y}_1(t) + \vec{Y}_2(t) \) is also a solution.
3. More generally, if \( \vec{Y}_1(t) \) and \( \vec{Y}_2(t) \) are two solutions of this system and \( k_1 \) and \( k_2 \) are any constants, then \( k_1 \vec{Y}_1(t) + k_2 \vec{Y}_2(t) \) is also a solution. A solution of the form \( k_1 \vec{Y}_1(t) + k_2 \vec{Y}_2(t) \) is called a linear combination of the solutions \( \vec{Y}_1(t) \) and \( \vec{Y}_2(t) \).
Linear independence

Definition (for functions):

The functions $y_1(t)$ and $y_2(t)$ are said to be *linearly dependent* on an interval $I$ if one of these functions is a constant multiple of the other on $I$. The functions $y_1(t)$ and $y_2(t)$ are said to be *linearly independent* on an interval $I$ if they are not linearly dependent on $I$. 
Linear independence

Definition (for vectors):

The vectors $\vec{y}_1$ and $\vec{y}_2$ are said to be linearly independent if they do not lie on the same line through the origin or, equivalently, if neither one is a constant multiple of the other.
Suppose we are to find the *general solution* for the IVP

\[
\frac{d\vec{Y}}{dt} = A\vec{Y}, \quad \vec{Y}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.
\]

Suppose further that we have somehow found two solutions, \( \vec{Y}_1(t) \) and \( \vec{Y}_2(t) \), of the linear system \( \frac{d\vec{Y}}{dt} = A\vec{Y} \).

Then we can find constants \( k_1 \) and \( k_2 \) so that \( k_1\vec{Y}_1(t) + k_2\vec{Y}_2(t) \)

is the *general solution* to the IVP \( \frac{d\vec{Y}}{dt} = A\vec{Y}, \quad \vec{Y}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \).
End Presentation