3.3 Phase Planes for Linear Systems with Real Eigenvalues
Consider the system: \[ \frac{d\vec{X}}{dt} = A\vec{X}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad T = a + d, \quad D = ad - bc. \]

With characteristic equation \( \lambda^2 - T\lambda + D = 0. \)

Recall that \( \lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}. \)

If the eigenvalues are real, we have \( T^2 - 4D \geq 0. \)
Unequal real eigenvalues

We consider all four subcases with eigenvalues $\lambda_1, \lambda_2$ and $\lambda_1 \neq \lambda_2$:

1. $\lambda_1 > \lambda_2 > 0$.

2. $\lambda_1 < \lambda_2 < 0$.

3. $\lambda_1 > 0 > \lambda_2$.

4. $\lambda_1 = 0, \lambda_2 \neq 0$. 
1. \( \lambda_1 > \lambda_2 > 0 \)

Example system:

\[
\frac{d\vec{Y}}{dt} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \vec{Y}
\]

with eigenvalues \( \lambda_1 = 4, \lambda_2 = 1 \). All nonzero solutions move away from the equilibrium point at the origin as \( t \to \infty \). The origin is a source.
\[ \frac{dx}{dt} = 2x + 2y \]
\[ \frac{dy}{dt} = x + 3y \]
2. $\lambda_1 < 0, \lambda_2 < 0$

Example system:

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \vec{Y}$$

with eigenvalues $\lambda_1 = -1, \lambda_2 = -4$. All nonzero solutions move toward the equilibrium point at the origin as $t \to \infty$. The origin is a sink.
\[
\frac{dx}{dt} = -x \\
\frac{dy}{dt} = -4y
\]
3. $\lambda_1 < 0, \lambda_2 > 0$

Example system:

$$\frac{d\bar{Y}}{dt} = \begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix} \bar{Y}$$

with eigenvalues $\lambda_1 = -5, \lambda_2 = 7$ and representative eigenvectors: $\bar{X}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \bar{X}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

The origin is a saddle.
\[
\frac{dx}{dt} = x + 12y
\]

\[
\frac{dy}{dt} = 3x + y
\]
4. $\lambda_1 = 0, \lambda_2 \neq 0$.

Example system:

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \vec{Y}$$

with eigenvalues $\lambda_1 = 0, \lambda_2 = -4$ and

representative eigenvectors: $\vec{X}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{X}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The system has a line of equilibrium points passing through the origin.
\[ \frac{dx}{dt} = -3x + y \]
\[ \frac{dy}{dt} = 3x - y \]
There is a line of equilibrium points given by \( y = 3x \), and every other solution approaches an equilibrium point on this line by following lines parallel to the line \( y = -x \).

Eigenvector \( \vec{X}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) lies on the line \( y = 3x \) and

eigenvector \( \vec{X}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \) lies on the line \( y = -x \).
In all these cases the general solution can be written in the form

\[ \vec{Y}(t) = k_1 e^{\lambda_1 t} \vec{X}_1 + k_2 e^{\lambda_2 t} \vec{X}_2 \]

where \( k_1 \) and \( k_2 \) are arbitrary constants and \( \vec{X}_1, \vec{X}_2 \) are representative eigenvectors for the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) respectively.
END