1) Use an integrating factor to find the explicit general solution $y(x)$ for

$$(1+x^2)\frac{dy}{dx} - 2xy = (1+x^2)^2$$

1) Solution

$$(1+x^2)\frac{dy}{dx} - 2xy = (1+x^2)^2 \Rightarrow \frac{dy}{dx} + \left(\frac{-2x}{1+x^2}\right)y = 1 + x^2$$

$$\Rightarrow \beta(x) = \exp\left(\int \left(-\frac{2x}{1+x^2}\right)dx\right) = \exp(-\ln(1+x^2)) = \frac{1}{1+x^2}$$

$$\Rightarrow \frac{d}{dx}\left[\frac{y}{1+x^2}\right] = 1 \Rightarrow \frac{y}{1+x^2} = x + C \Rightarrow y(x) = (x+C)(1+x^2)$$
2) Consider the following 8 first-order equations:

1. \( \frac{dy}{dt} = 9 - y^2 \)
2. \( \frac{dy}{dt} = 9 - t^2 \)
3. \( \frac{dy}{dt} = y^2 - 9 \)
4. \( \frac{dy}{dt} = 1 - 2y \)
5. \( \frac{dy}{dt} = 2y(1-y) \)
6. \( \frac{dy}{dt} = 4y^2 \)
7. \( \frac{dy}{dt} = -4y^2 \)
8. \( \frac{dy}{dt} = t - y \)

Four of the associated slope fields are shown on the next page. Pair the slope fields with their associated equations. Justify your choices - no credit unless you justify your selections.

a.) The equation for Slope Field A is __2__. My reason for choosing this answer is:

b.) The equation for Slope Field B is __7__. My reason for choosing this answer is:

c.) The equation for Slope Field C is __3__. My reason for choosing this answer is:

d.) The equation for Slope Field D is __4__. My reason for choosing this answer is:
Here are the equations again for reference:

1. \[ \frac{dy}{dt} = 9 - y^2 \]
2. \[ \frac{dy}{dt} = 9 - t^2 \]
3. \[ \frac{dy}{dt} = y^2 - 9 \]
4. \[ \frac{dy}{dt} = 1 - 2y \]
5. \[ \frac{dy}{dt} = 2y(1 - y) \]
6. \[ \frac{dy}{dt} = 4y^2 \]
7. \[ \frac{dy}{dt} = -4y^2 \]
8. \[ \frac{dy}{dt} = t - y \]
3) Given the IVP \( \frac{dy}{dt} = \cos t, \quad y(0) = 0, \)

a) use Euler's method with \( \Delta t = \pi/10 \) to approximate \( y(\pi/2). \)

Fill in the table below with your approximate values to three decimal places.

b) Find the exact solution for the differential equation.

c) Compare the values you found in a) with values given by the solution in b).

What is the absolute value of the error at \( y(\pi/2)? \)

<table>
<thead>
<tr>
<th>Step number ( k )</th>
<th>( t_k )</th>
<th>( y_k )</th>
<th>Exact ( y(t_k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{\pi}{10} \approx 0.314159265 )</td>
<td>0.314159265359</td>
<td>0.30901699437</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{\pi}{5} \approx 0.628318530 )</td>
<td>0.612942481833</td>
<td>0.587785252224</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{3\pi}{10} \approx 0.942477796 )</td>
<td>0.867102666449</td>
<td>0.809016994374975</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{2\pi}{5} \approx 1.2566370615 )</td>
<td>1.0517608495</td>
<td>0.951056516295153</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{\pi}{2} \approx 1.570796326 )</td>
<td>1.14884140143</td>
<td>1.000</td>
</tr>
</tbody>
</table>

\[
y_1 = y_0 + \frac{\pi}{10} \cos t_0 = 0 + \frac{\pi}{10} \cos (0) = \frac{\pi}{10} \approx 0.314159265359
\]

\[
y_2 = y_1 + \frac{\pi}{10} \cos t_1 = 0.314159265359 + \frac{\pi}{10} \cos \left( \frac{\pi}{10} \right) \approx 0.612942481833
\]

\[
y_3 = y_2 + \frac{\pi}{10} \cos t_2 = 0.612942481833 + \frac{\pi}{10} \cos \left( \frac{\pi}{5} \right) \approx 0.867102666449
\]

\[
y_4 = y_3 + \frac{\pi}{10} \cos t_3 = 0.867102666449 + \frac{\pi}{10} \cos \left( \frac{3\pi}{10} \right) \approx 1.0517608495
\]

\[
y_5 = y_4 + \frac{\pi}{10} \cos t_4 = 1.0517608495 + \frac{\pi}{10} \cos \left( \frac{2\pi}{5} \right) \approx 1.14884140143
\]

A simple integration tells us that \( y(t) = \sin t + C \) and \( y(0) = 0 \) implies that

\( C = 0 \). So \( y(t) = \sin t \) and \( y\left( \frac{\pi}{2} \right) = 1 \). The absolute error of our approximation is

\[ |1 - 1.14884140143| = 0.14884140143 \]
4) Use the method of guessing to find the general solution for the differential equation:

\[ \frac{dy}{dx} = \frac{y}{2} + 4 \exp \left( \frac{t}{2} \right) \]

The general solution of the associated homogeneous equation is \( y_h(t) = ke^{t/2} \). For a particular solution of the nonhomogeneous equation, we guess \( y_p(t) = \alpha te^{t/2} \) rather than \( \alpha e^{t/2} \) because \( \alpha e^{t/2} \) is a solution of the homogeneous equation. Then

\[
\frac{dy_p}{dt} - \frac{y_p}{2} = \alpha e^{t/2} + \frac{\alpha e^{t/2}}{2} - \frac{\alpha e^{t/2}}{2} = \alpha e^{t/2}.
\]

Consequently, we must have \( \alpha = 4 \) for \( y_p(t) \) to be a solution. Hence, the general solution to the nonhomogeneous equation is

\[ y(t) = ke^{t/2} + 4te^{t/2}. \]
5) Consider the population model \( \frac{dP}{dt} = P^5 \left( 1 - \frac{P}{15} \right) \left( \frac{P}{7} - 1 \right) \) with \( P(0) = 3 \).

where \( P(t) \) is the population at time \( t \).

a) Use a phase portrait for the equation to give a rough sketch of the solution \( P(t) \) satisfying the initial condition.

b) Since the initial condition falls in the interval \( 0 < P < 7 \), we see that \( P(t) \to 0 \) as \( t \) becomes very large.
6) Locate the bifurcation value(s) for the one-parameter family $\frac{dx}{dt} = 1 + cx + x^2$, where $c$ is a real number, and sketch the complete bifurcation diagram of equilibrium solutions against $c$. Label your graph appropriately and completely.

![Bifurcation Diagram](image)

Locate the equilibria by setting $\frac{dx}{dt} = 1 + cx + x^2 = 0$, which gives

$$x = \frac{-c \pm \sqrt{c^2 - 4}}{2}.$$

If $c^2 - 4 < 0$ $\Rightarrow -2 < c < 2$. There are no equilibria.

If $c^2 - 4 = 0$ $\Rightarrow c = \pm 2$. There are two equilibria.

When $c = 2$, $x = -1$ and when $c = -2$, $x = 1$.

If $c^2 - 4 > 0$ $\Rightarrow c > 2$ or $c < -2$. There are again two equilibria.
So the bifurcation values are \( c = \pm 2 \).

Note that with \( f'(x) = 1 + cx + x^2 \), \( f''(x) = c + 2x \) and apply the Derivative Test to determine the nature of the equilibrium solutions:

When \( c > 2 \) or \( c < -2 \),

\[
f'' \left( \frac{-c \pm \sqrt{c^2 - 4}}{2} \right) = c + 2 \left( \frac{-c \pm \sqrt{c^2 - 4}}{2} \right)
\]

\[= \pm \sqrt{c^2 - 4}, \text{ so that the equilibrium solution with the positive square root is a source and the equilibrium solution with the negative square root is a sink.} \]

If \( c = \pm 2 \), there is a node. In this case the Derivative Test fails and you must determine how the derivative behaves separately.
7) Find the general solution for the differential equation

\[ \frac{dy}{dx} = \frac{(y-1)(y-2)}{x}. \]

Solution:
The equation is separable so write

\[ \frac{dy}{(y-1)(y-2)} = \frac{dx}{x}. \]

Then integrate giving

\[ \int \frac{dy}{(y-1)(y-2)} = \int \frac{dx}{x} \]

First, consider the integral on the left hand side. This form is best handled by the method of partial fractions. Set

\[ \frac{1}{(y-1)(y-2)} = \frac{A}{y-1} + \frac{B}{y-2} \]

with constants \( A \) and \( B \) to be determined.

Clearing fractions we have

\[ A(y-2) + B(y-1) = 1 \]

Collecting like terms we get

\[ (A+B)y - 2A - B = 0 \cdot y + 1 \]

We must have

\[ A + B = 0 \Rightarrow B = -A \text{ and then } -2A - B = 1 = -2A + A \Rightarrow A = -1, B = 1 \]

and so

\[ \int \frac{dy}{(y-1)(y-2)} = \int \left( \frac{1}{y-2} - \frac{1}{y-1} \right) dy = \ln|y-2| - \ln|y-1|. \]

On the right hand side we have

\[ \int \frac{dx}{x} = \ln|x| + C_1 \]

So

\[ \ln|y-2| - \ln|y-1| = \ln|x| + C_1 \Rightarrow \ln \left| \frac{y-2}{y-1} \right| = \ln|x| + C_1 \]

\[ \Rightarrow \left| \frac{y-2}{y-1} \right| = C_2 |x| \Rightarrow \frac{y-2}{y-1} = Cx \Rightarrow y = \frac{2-Cx}{1-Cx} \]

The general solution is

\[ y(x) = \frac{2-Cx}{1-Cx} \]
8) Use the method of guessing to find the general solution for the differential equation:

\[
\frac{dy}{dx} = 2y + \sin 2x
\]

Solution:

First solve the homogeneous equation \( \frac{dy}{dx} - 2y = 0 \) which is separable.

The solution is \( y_h(x) = ke^{2x} \).

Now to find a particular solution to the nonhomogeneous equation, make the guess \( y_p(x) = \alpha \sin 2x + \beta \cos 2x \), with \( \alpha, \beta \) constants to be determined.

Differentiating \( y_p(x) \) gives \( y'_p(x) = 2\alpha \cos 2x - 2\beta \sin 2x \)

Substituting into the original equation gives

\[
\frac{dy_p}{dx} - 2y_p = 2\alpha \cos 2x - 2\beta \sin 2x - 2(\alpha \sin 2x + \beta \cos 2x)
\]

\[= (2\alpha - 2\beta) \cos 2x + (-2\beta - 2\alpha) \sin 2x.\]

Now for \( y_p(x) \) to be a solution, we must have

\[
(2\alpha - 2\beta) \cos 2x + (-2\beta - 2\alpha) \sin 2x = \sin 2x \Rightarrow
\]

\[
2\alpha - 2\beta = 0 \quad \Rightarrow \quad \alpha = \beta \quad \text{and} \quad \alpha = -\frac{1}{4}, \beta = -\frac{1}{4}.
\]

The general solution is then

\[
y(x) = y_h(x) + y_p(x) = ke^{2x} - \frac{1}{4} \sin 2x - \frac{1}{4} \cos 2x
\]
9) Convert the second order equation \( \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 20y = 0 \) to an equivalent first order system (do not try to solve the system!). Be sure to write down the first order system completely.

To find the equivalent first order linear system, introduce the new variable

\[ v = \frac{dy}{dx} \Rightarrow \frac{dv}{dx} = \frac{d^2 y}{dx^2}. \]

Now solve the differential equation for \( \frac{d^2 y}{dx^2} \) to get

\[ \frac{d^2 y}{dx^2} = 4 \frac{dy}{dx} - 20y \text{ or } \frac{d^2 y}{dx^2} = 4v - 20y \Rightarrow \frac{dv}{dx} = 4v - 20y. \]

The equivalent first order system is thus:

\[
\begin{align*}
\frac{dy}{dx} &= v \\
\frac{dv}{dx} &= 4v - 20y
\end{align*}
\]

or, in matrix form,

\[
\begin{bmatrix}
\frac{dy}{dx} \\
\frac{dv}{dx}
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -20 & 4 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}.
\]