Sinusoidal Forcing

We now move to equations of the form

$$\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + qy = g(t),$$

where $g(t)$ is a sine or cosine function.

These external forces, $g(t)$, have the following properties:

1. They are periodic. Let $T$ be some fixed length of time. These forces repeat after time $T$.
   That is, $g(t + T) = g(t)$ for all $t$.

2. They are of zero average over one full period.
   That is, $\int_0^T g(t) \, dt = 0$. 
Suppose we have the equation

\[
\frac{d^2y}{dt^2} + 2 \frac{dy}{dt} + 2y = \sin t
\]

To find a particular solution, the Method of Undetermined Coefficients suggests we use a guess \(y_p(t) = A \sin t + B \cos t\). We have been using exponential guesses all along, and it is more efficient if we do so here as well.

To that end, we consider the complex differential equation

\[
\frac{d^2y}{dt^2} + 2 \frac{dy}{dt} + 2y = \exp(it)
\]
The solution of the complex differential equation will have real and imaginary parts. It looks like

\[ y_C(t) = y_{RE}(t) + i \cdot y_{IM}(t) \]

Plugging into the differential equation ...

\[ \frac{d^2 (y_{RE} + i \cdot y_{IM})}{dt^2} + 2 \frac{d (y_{RE} + i \cdot y_{IM})}{dt} + 2(y_{RE} + i \cdot y_{IM}) = \cos t + i \sin t \]

Splitting into real and imaginary parts, we have

\[ \frac{d^2 y_{RE}}{dt^2} + 2 \frac{dy_{RE}}{dt} + 2y_{RE} = \cos t \]

and

\[ \frac{d^2 y_{IM}}{dt^2} + 2 \frac{dy_{IM}}{dt} + 2y_{IM} = \sin t \]
So the imaginary part, \( y_{IM}(t) \), is a particular solution of the original equation with the forcing function \( \sin t \).

The main advantage of using the complex equation approach is that the appropriate guess for the particular solution is once again another exponential.

So for the equation \( \frac{d^2 y}{dt^2} + 2\frac{dy}{dt} + 2y = \exp(it) \), we guess \( y_C(t) = a\exp(it) \)

and solve for the complex constant \( a \).
Plugging into the differential equation ...

\[
\frac{d^2 y_c}{dt^2} + 2 \frac{dy_c}{dt} + 2y_c = -a \exp(it) + 2ai \exp(it) + 2a \exp(it)
\]

\[
= a(1+2i)\exp(it)
\]

For \( y_c \) to be a solution we must have

\[
a(1+2i)\exp(it) = \exp(it) \Rightarrow a = \frac{1}{1+2i} = \frac{1-2i}{5}.
\]
To determine the real and imaginary parts, we write

\[ y_c(t) = \frac{1-2i}{5}(\cos t + i \sin t) = \left( \frac{1}{5} \cos t + \frac{2}{5} \sin t \right) + i \left( \frac{-2}{5} \cos t + \frac{1}{5} \sin t \right) \]

Taking the imaginary part, we get a particular solution of the original equation with forcing function \( \sin t \).

\[ y_p(t) = -\frac{2}{5} \cos t + \frac{1}{5} \sin t \]

and the general solution is thus

\[ y(t) = k_1 \exp(-t) \sin t + k_2 \exp(-t) \cos t - \frac{2}{5} \cos t + \frac{1}{5} \sin t \]
End Presentation