3.4 Complex eigenvalues

In this case we have eigenvalues $\lambda = \alpha \pm \beta i$ with α, β real numbers and $i = \sqrt{-1}$. The λ's form a complex conjugate pair. The behavior of trajectories in this case depends on the *real part*, α, of the complex eigenvalues. When the eigenvalues are complex, the eigenvectors will also have complex entries.

Even when the matrix A has complex eigenvalues, the general solution of $\dfrac{d\vec{X}}{dt} = A\vec{X}$ has the form $\vec{X}(t) = c_1 e^{\lambda_1 t}\vec{X}_1 + c_2 e^{\lambda_2 t}\vec{X}_2$.

We will need Euler's formula again:

$$e^{\alpha+\beta i} = e^{\alpha} e^{\beta i} = e^{\alpha}\left[\cos\beta + i\sin\beta\right].$$

This formula is useful for simplifying complex-valued expressions and will help to obtain real-valued solutions. of $\vec{X}' = A\vec{X}$.

Another important fact is that *eigenvectors corresponding to complex conjugate eigenvalues are conjugate to each other.*

If the eigenvalue $\lambda_1 = \alpha + \beta i$ has a corresponding eigenvector

$$\vec{V}_1 = \begin{bmatrix} a_1 + b_1 i \\ a_2 + b_2 i \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + i \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \vec{U} + i\vec{W},$$

then $\lambda_2 = \overline{\lambda_1} = \alpha - \beta i$ has a corresponding eigenvector

$$\vec{V}_2 = \overline{\vec{V}_1} = \begin{bmatrix} a_1 - b_1 i \\ a_2 - b_2 i \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - i \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \vec{U} - i\vec{W}.$$

Suppose $\lambda = \alpha + \beta i$ is an eigenvalue for the matrix A and that $\vec{V} = \vec{U} + i\vec{W}$ is a corresponding eigenvector.

If we define $\vec{X}(t) = e^{\lambda t}\vec{V}$, then $A\vec{X} = A\left(e^{\lambda t}\vec{V}\right) = e^{\lambda t}\left(A\vec{V}\right)$

$$= e^{\lambda t}\left(\lambda\vec{V}\right) = \lambda e^{\lambda t}\vec{V} = \vec{X}',$$

so $\vec{X}(t)$ is a solution of the system.

By Euler we have

$$\vec{X}(t) = e^{\lambda t}\vec{V} = e^{(\alpha+\beta i)t}\vec{V} = e^{\alpha t}(\cos\beta t + i\sin\beta t)\left(\vec{U} + i\vec{W}\right)$$

$$= e^{\alpha t}\left\{(\cos\beta t)\vec{U} - (\sin\beta t)\vec{W}\right\} + ie^{\alpha t}\left\{(\cos\beta t)\vec{W} + (\sin\beta t)\vec{U}\right\}$$

Then the *real part* and the *imaginary part* of $\vec{X}(t)$ can be considered separately.

$$\vec{X}_1(t) = \mathrm{Re}\left\{\vec{X}(t)\right\} = e^{\alpha t}\left\{(\cos\beta t)\vec{U} - (\sin\beta t)\vec{W}\right\}$$

$$\vec{X}_2(t) = \mathrm{Im}\left\{\vec{X}(t)\right\} = e^{\alpha t}\left\{(\cos\beta t)\vec{W} + (\sin\beta t)\vec{U}\right\}$$

It is important to note that $\vec{X}_1(t)$ and $\vec{X}_2(t)$ are *real-valued linearly independent solutions of the system* $\vec{X}' = A\vec{X}$.

Also, by the superposition principle, we know that $c_1\vec{X}_1(t) + c_2\vec{X}_2(t)$ is also a solution.

Example: A system with complex eigenvalues.

Consider the system $\dfrac{dx}{dt} = y, \quad \dfrac{dy}{dt} = -k^2 x$ or in matrix form $\begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ with characteristic equation

$\lambda^2 + k^2 = 0$ and complex conjugate eigenvalues

$\lambda_1 = ki, \quad \lambda_2 = -ki.$

The equation $A\vec{X} = \lambda_1 \vec{X}$ has the form $\begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ki\,x \\ ki\,y \end{bmatrix}$

and $\begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kix \\ kiy \end{bmatrix}$ is equivalent to the algebraic system

$$\begin{aligned} y &= kix \\ -k^2 x &= kiy \end{aligned}$$

where the second equation is just ki times the first. We can take x as arbitrary and $y = kix$ and we get the eigenvector $\vec{V} = \begin{bmatrix} x \\ kix \end{bmatrix} = x\begin{bmatrix} 1 \\ ki \end{bmatrix}$. Pick $x = 1$ and we get the representative eigenvector $\vec{V}_1 = \begin{bmatrix} 1 \\ ki \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ k \end{bmatrix}$.

We don't have to worry about the second (conjugate) eigenvalue and its associated eigenvector. The general solution of our original system can be obtained from the information we now have available. We start with the complex solution

$$\vec{X}(t) = e^{kit}\left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\k\end{bmatrix}\right) = (\cos kt + i\sin kt)\left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\k\end{bmatrix}\right)$$

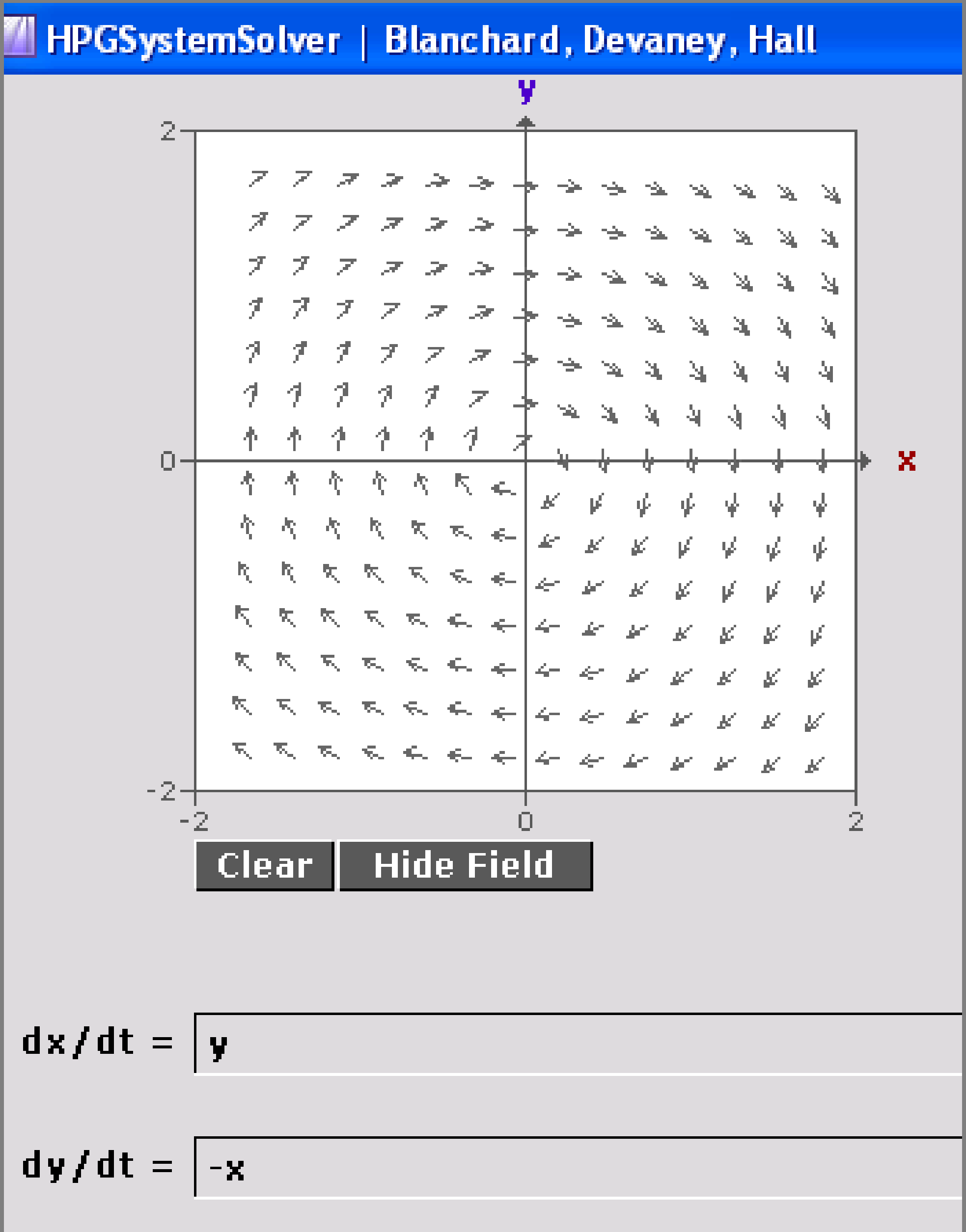
$$= \left\{(\cos kt)\begin{bmatrix}1\\0\end{bmatrix} - (\sin kt)\begin{bmatrix}0\\k\end{bmatrix}\right\} + i\left\{(\cos kt)\begin{bmatrix}0\\k\end{bmatrix} + (\sin kt)\begin{bmatrix}1\\0\end{bmatrix}\right\}$$

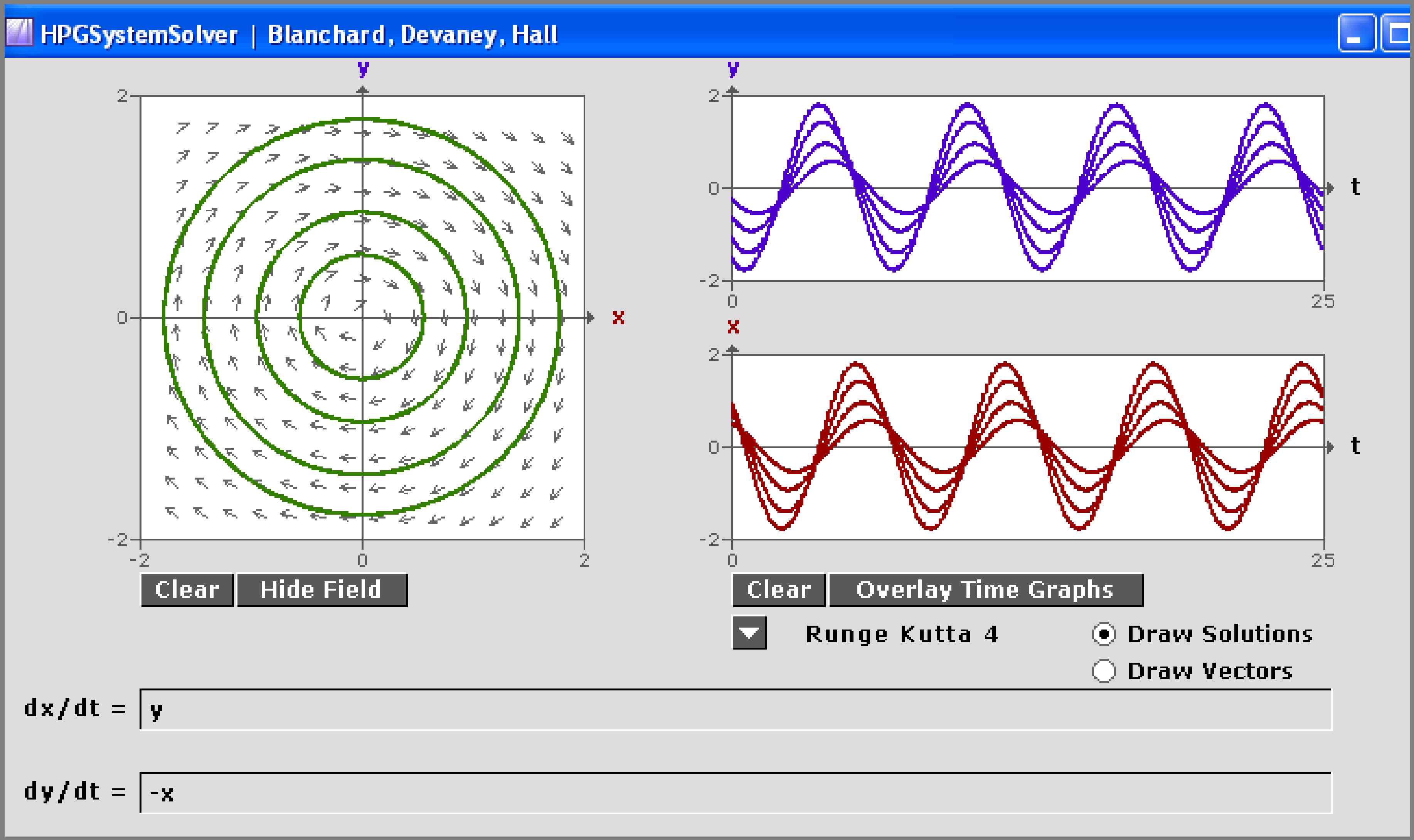
Because the real and imaginary parts of the last expression are linearly independent solutions of the system, the general solution is given by

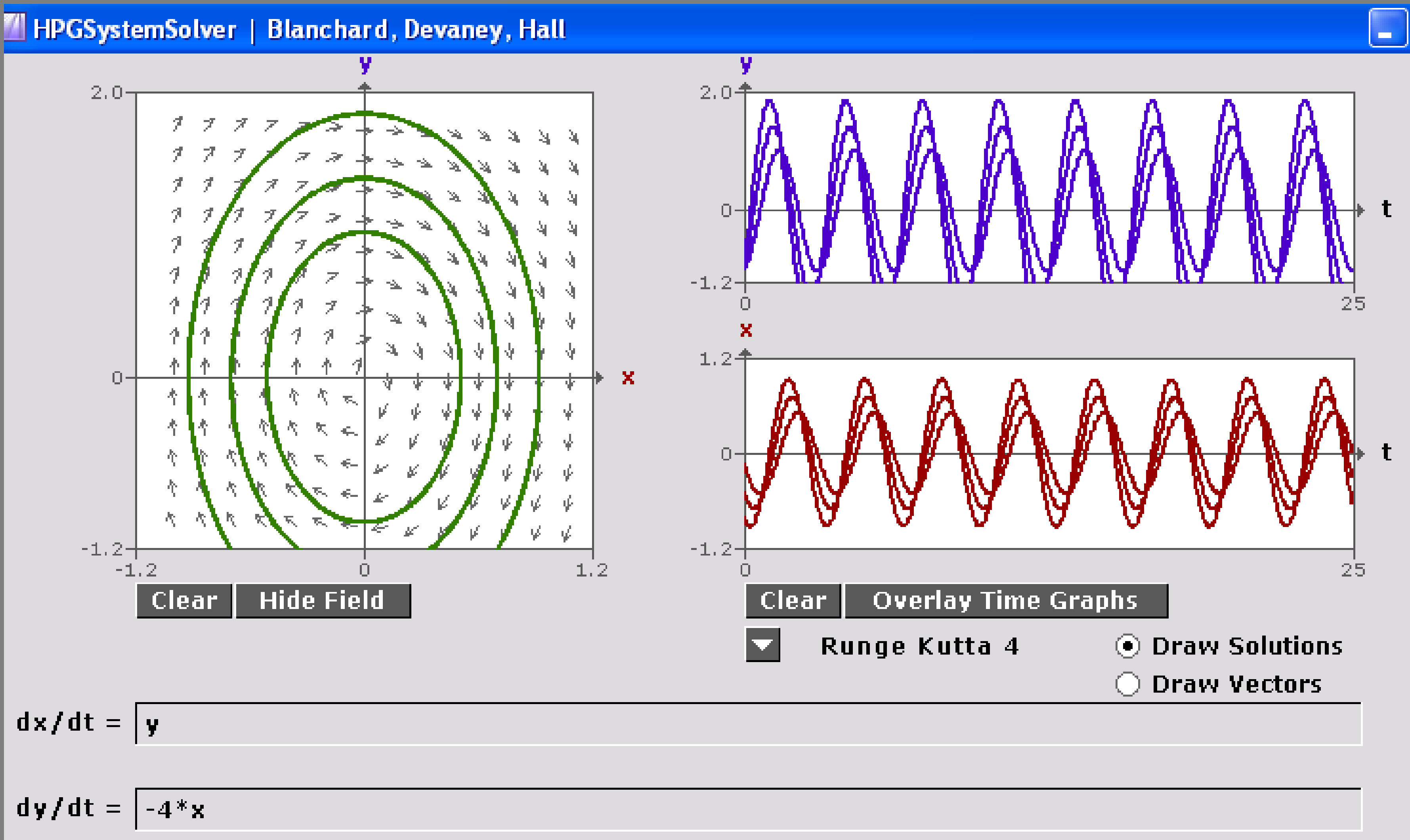
$$\vec{X}(t) = c_1 \left\{ (\cos kt) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - (\sin kt) \begin{bmatrix} 0 \\ k \end{bmatrix} \right\} + c_2 \left\{ (\cos kt) \begin{bmatrix} 0 \\ k \end{bmatrix} + (\sin kt) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

$$= c_1 \begin{bmatrix} \cos kt \\ -k \sin kt \end{bmatrix} + c_2 \begin{bmatrix} \sin kt \\ -k \cos kt \end{bmatrix} = \begin{bmatrix} c_1 \cos kt + c_2 \sin kt \\ -k c_1 \sin kt + k c_2 \cos kt \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The equilibrium point at the origin is a *center*.


HPGSystemSolver | Blanchard, Devaney, Hall
y
2
0
-2
x
-2
0
2
Clear
Hide Field
dx/dt = y
dy/dt = -x


HPGSystemSolver | Blanchard, Devaney, Hall
y
x
t
2
0
-2
25
Clear
Hide Field
Clear
Overlay Time Graphs
Runge Kutta 4
Draw Solutions
Draw Vectors
dx/dt = y
dy/dt = -x


HPGSystemSolver | Blanchard, Devaney, Hall
y
2.0
0
-1.2
-1.2
1.2
x
t
25
Clear
Hide Field
Clear
Overlay Time Graphs
Runge Kutta 4
Draw Solutions
Draw Vectors
dx/dt =
y
dy/dt =
-4*x

In our example the eigenvalues were purely imaginary. In general, we will get eigenvalues with nonzero real and imaginary parts; $\lambda = \alpha \pm \beta i$. Then our (complex) solutions take the form

$$\vec{X}(t) = e^{\lambda t} = e^{(\alpha \pm \beta i)t} = e^{\alpha t} e^{\beta i t} = e^{\alpha t}(\cos \beta t + i \sin \beta t).$$

Then $\mathrm{Re}\left(\vec{X}(t)\right) = e^{\alpha t} \cos \beta t$ and $\mathrm{Im}\left(\vec{X}(t)\right) = e^{\alpha t} \sin \beta t$ are two linearly independent real solutions of the system and the general solution is then

$$\vec{X}(t) = k_1 e^{\alpha t} \cos \beta t + k_2 e^{\alpha t} \sin \beta t, \; k_1, k_2 \text{ arbitrary constants.}$$

Or the general solution can be written

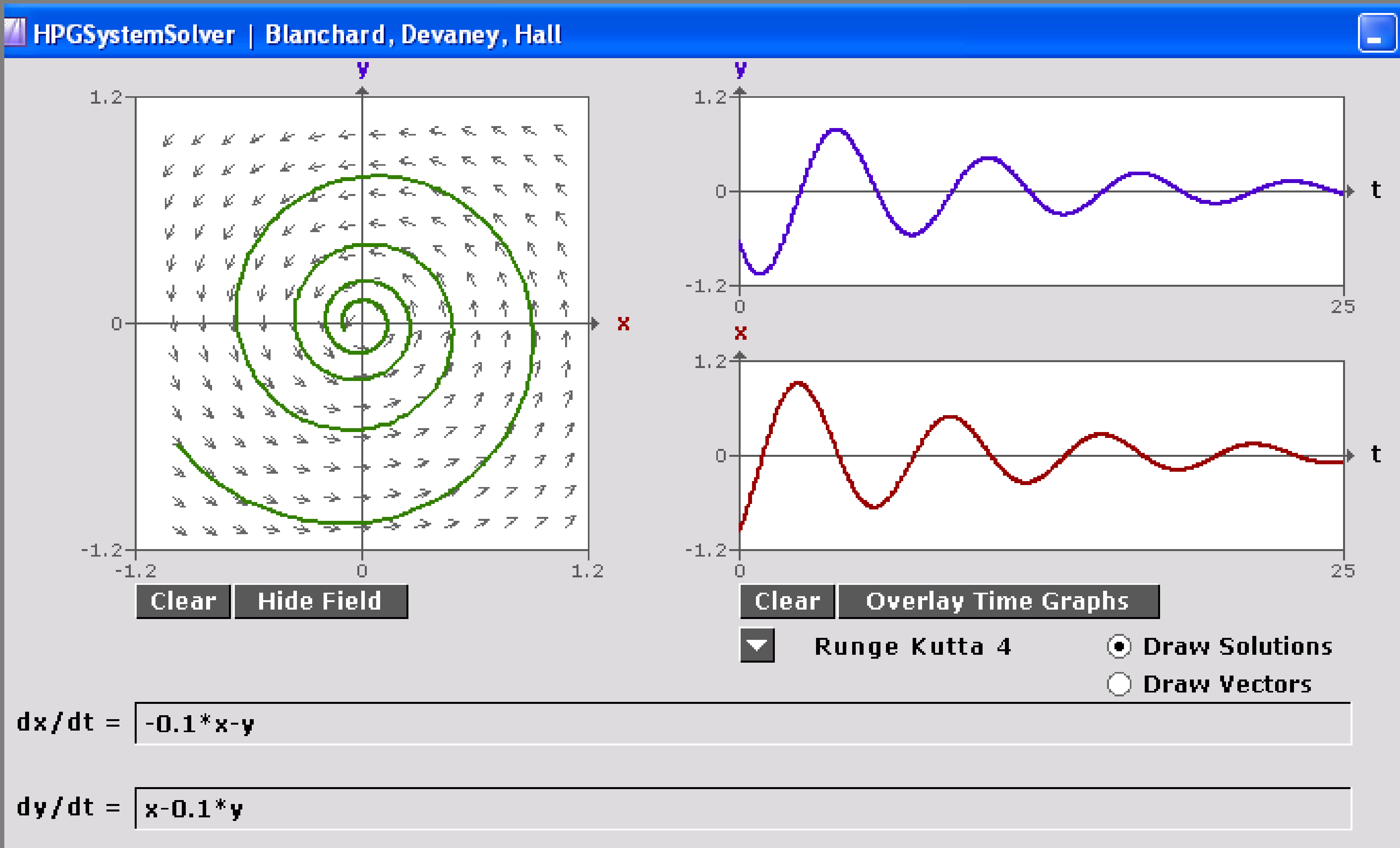
$$\vec{X}(t) = e^{\alpha t}\left(k_1 \cos \beta t + k_2 \sin \beta t\right).$$

If $\operatorname{Re}(\lambda) = \alpha$ is such that

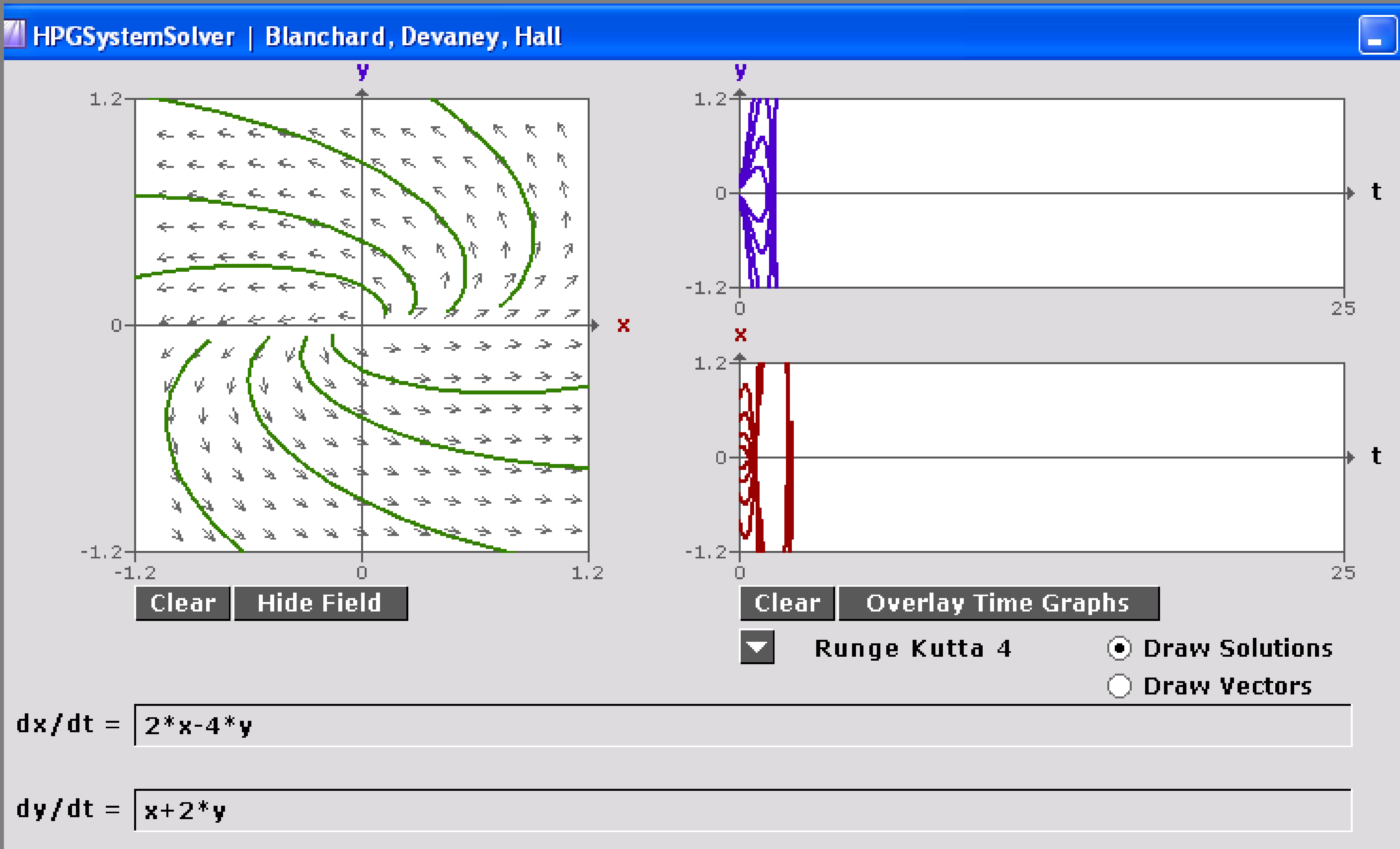
$\alpha > 0 \quad \Rightarrow \quad$ the origin is a *spiral source*.

$\alpha < 0 \quad \Rightarrow \quad$ the origin is a *spiral sink*.

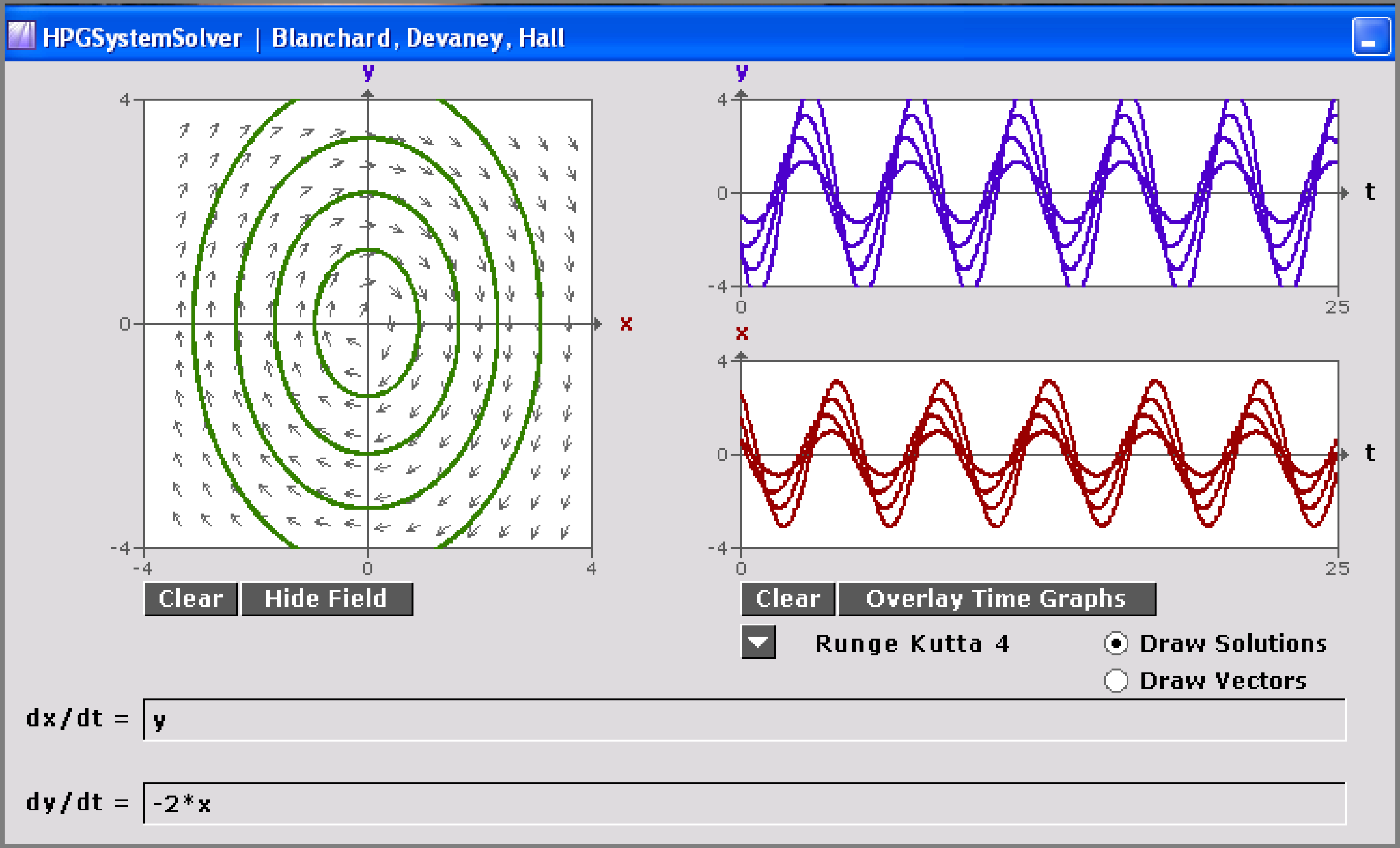
$\alpha = 0 \quad \Rightarrow \quad$ the origin is a *center*.


HPGSystemSolver | Blanchard, Devaney, Hall
y
x
1.2
0
-1.2
-1.2
0
1.2
t
25
Clear
Hide Field
Clear
Overlay Time Graphs
Runge Kutta 4
Draw Solutions
Draw Vectors
dx/dt = -0.1*x-y
dy/dt = x-0.1*y

Spiral sink



Spiral source



HPGSystemSolver | Blanchard, Devaney, Hall
y
x
t
Clear
Hide Field
Clear
Overlay Time Graphs
Runge Kutta 4
Draw Solutions
Draw Vectors
dx/dt = y
dy/dt = -2*x

Center

Summary of Stability Criteria for 2-D Linear Systems

Eigenvalues	Stability
REAL	
Unequal	
Both > 0	Unstable Node (source, repeller)
Both < 0	Stable Node (sink, attractor)
Different signs	Saddle Point
One $= 0$, the other $\neq 0$	Whole line of equilibrium points

Summary of Stability Criteria for 2-D Linear Systems

Eigenvalues	Stability
REAL	
Equal	
Both > 0	Unstable Node (source, repeller)
Both < 0	Stable Node (sink, attractor)
Both = 0	Algebraically unstable

Summary of Stability Criteria for 2-D Linear Systems

Eigenvalues	Stability
COMPLEX	
Real part > 0	Spiral source (unstable spiral, repeller)
Real part < 0	Spiral sink (stable spiral)
Real part = 0	Center (stable center)

End Presentation