Work of a Force

- Vector \( \mathbf{dr} \) is an infinitesimal displacement.
- Work done by force
  \[
  dW = F \cdot \mathbf{dr} = F(r_x \, dx + r_y \, dy + r_z \, dz)
  \]
- Work is a scalar quantity, i.e., it has magnitude and sign but not direction.
- Units of work:
  \[
  1 \text{ joule} = 1 \text{ N} \cdot \text{m} = 1 \text{ lb} \cdot \text{ft}
  \]

Work of a Constant Force

- Constant force in rectilinear motion,
  \[
  U_{1\to2} = \int_{x_1}^{x_2} \mathbf{F} \cdot \mathbf{ds} = \int_{x_1}^{x_2} F_\alpha \cos \alpha \, ds
  \]
- Force of gravity,
  \[
  dW = -W \, dy
  \]
  \[
  U_{1\to2} = -\int_{y_1}^{y_2} W \, dy = -W (y_2 - y_1) = -W \Delta y
  \]
- Positive when the particle moves down.

Work of a Spring Force

- Spring force:
  \[
  F_s = -kx
  \]
- Work of the force exerted by spring,
  \[
  dU = -F \, dx = -kx \, dx
  \]
- Work of the spring force is positive when \( |x_2| < |x_1| \), i.e., when the spring is returning to its undeformed position.

Work of Gravitational Force

Consider two particles \( M \) at position \( O \) and particle \( m \) moving along the path shown.

\[
\begin{align*}
  dU &= -G \frac{Mm}{r^2} \, dr \\
  U_{1\to2} &= -\int_{r_1}^{r_2} G \frac{Mm}{r^2} \, dr = G \frac{Mm}{r_2} - G \frac{Mm}{r_1}
\end{align*}
\]
Work-Energy Principle

Derivation:

\[ F_x = m a_x \]
\[ F_x = m \frac{dv}{dt} \]
\[ F_x = m \frac{d^2x}{dt^2} \]
\[ F_x = m a_x = \frac{F}{m} \]
\[ a_x = \frac{dv}{dt} \]
\[ a_x = \frac{d^2x}{dt^2} \]
\[ \int F_x dx = \frac{1}{2} m v^2 - \frac{1}{2} m v_i^2 \]

\[ U_{1\rightarrow 2} = T_a - T_i \]

- The work done by forces acting on a particle is equal to the change in kinetic energy of the particle.

Power and Efficiency

- Power = rate at which work is done.

\[ P = \frac{dU}{dt} \]
\[ P = F \cdot v \]

- Dimensions of power are work/time or force*velocity.

Units for power are

1 W (watt) = \( \frac{1}{2} \) N \( \frac{m}{s} \) or 1 hp = 550 \( \frac{lb}{ft} \) = 746 W

- \( \eta \) = efficiency

\[ \eta = \frac{\text{output work}}{\text{input work}} \]
\[ \eta = \frac{\text{power output}}{\text{power input}} \]

Example 1

Two blocks are joined by an inextensible cable as shown. If the system is released from rest, determine the velocity of block A after it has moved 2 m. Assume that the coefficient of friction between block A and the plane is \( \mu_k = 0.25 \) and that the pulley is weightless and frictionless.

\[ A: \quad U_{1\rightarrow 2} = \frac{1}{2} m A v_f^2 - \frac{1}{2} m A v_i^2 \]
\[ U_{1\rightarrow 2} = U_f + U_r \]
\[ = T_c g (2m) - F_r (xm) \]
\[ T_c g (2m) = T \cdot m A g \]
\[ = \frac{1}{2} m A v_f^2 \]

\[ B: \quad U_{1\rightarrow 2} = \frac{1}{2} m B v_f^2 - \frac{1}{2} m B v_i^2 \]
\[ U_{1\rightarrow 2} = U_f + U_r \]
\[ = m B g (2m) - T (2m) \]
\[ = \frac{1}{2} m B v_f^2 \]

Example 2. The 100-lb block slides down the inclined plane (\( \mu_k = 0.25 \)). The spring is of constant \( k = 200 \text{lb/ft} \).

If \( v_A = 10 \text{ ft/s} \), determine the maximum deformation of the spring.

\[ U_{A\rightarrow B} = \frac{1}{2} m A v_f^2 + \frac{1}{2} m B v_f^2 - \frac{1}{2} m B v_i^2 \]

\[ U_{A\rightarrow B} = U_A + U_f + U_r \]

N - W(xf) = 0

\[ x_f = \frac{N}{W} \]

\[ U_{A\rightarrow B} = U_A + U_f + U_r \]

\[ U_f = \frac{1}{2} k x_f^2 \]

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Potential Energy: Gravity

\[ W = \int F \cdot dx = \int \frac{GMm}{r^2} \cdot dx \]

Potential energy \( V_g \) when the variation in the force of gravity can not be neglected,

\[ V_g = \frac{GMm}{r} \]

Conservative Forces

\[ W = \int F \cdot dx = \int \frac{\partial V}{\partial x} \cdot dx = \frac{1}{2} \left( \frac{\partial V}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial V}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial V}{\partial z} \right)^2 \]

Conservation of Energy

\[ U_{1-2} = T_1 - T_2 \]

If all forces acting are conservative,

\[ U_{1-2} = V_1 - V_2 \]

Conservation of energy:

\[ T_1 + V_1 = T_2 + V_2 \]

Work of the force of gravity \( W \),

\[ W = \frac{1}{2} \left( \frac{\partial V}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial V}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial V}{\partial z} \right)^2 \]

Units of work and potential energy are the same: \( J \cdot m \)
Example 3.

The 0.5 lb pellet is pushed against the spring and released from rest at A. Neglecting friction, determine the smallest deflection of the spring for which the pellet will travel around the loop and remain in contact with the loop at all times.

Motion Under a Conservative Central Force

• When a particle moves under a conservative central force, both the principle of conservation of angular momentum

\[ \frac{1}{2} k x^2 = \frac{1}{2} m v^2 + mg (1 - \cos \theta) \]

and the principle of conservation of energy

\[ \frac{1}{2} k x^2 = \frac{1}{2} m v^2 + \frac{1}{2} m v^2 - \frac{G m m}{r} \]

may be applied.

• Given r, the equations may be solved for v and \( \phi \).

• At minimum and maximum r, \( \phi = 90^\circ \). Given the launch conditions, the equations may be solved for \( r_{\text{min}} \), \( r_{\text{max}} \), \( v_{\text{min}} \), and \( v_{\text{max}} \).

Example 4

A satellite is launched in a direction parallel to the surface of the earth with a velocity of 36900 km/h from an altitude of 500 km.

Determine (a) the maximum altitude reached by the satellite, and (b) the position of the satellite when its speed is 35000 km/h.