Problems

You may use a calculator to verify solutions, but not to provide them.

1. 900 cc/sec
   Hint: \( \frac{dx}{dt} = 3 \text{ cm/sec.} \), want \( \frac{dV}{dt} \), and we know \( V = x^3 \) so \( \frac{dV}{dx} = 3x^2 \)

2. \( \frac{1}{36} \) m/min or \( \frac{10}{3} \) cm/min.
   Hint: Want \( \frac{dh}{dt} \) and we know \( \frac{dV}{dt} = 15 \text{ m}^3/\text{min.} \).
   The volume of the pool for \( h \leq 2 \) is \( V = \frac{1}{2} \pi \cdot h \cdot x \).
   From the triangle, we have \( \frac{x}{h} = \frac{\frac{30}{2}}{\frac{15}{2}} \rightarrow x = 15h \).
   So \( V = \frac{225}{2} h^2 \) and \( \frac{dV}{dh} = 225h \rightarrow = 450 \) at \( h = 2 \).

3. Draining at \( \frac{49\pi}{35} \) ft/min.
   Hint: Want \( \frac{dV}{dt} \) and we know \( \frac{dh}{dt} = -0.25 \) ft/min.
   To relate \( V \) and \( h \) we know \( V = \frac{1}{3} \pi r^2 h \) but we need to eliminate the \( r \) so use similar triangles: \( \frac{r}{h} = \frac{\frac{7}{12}}{\frac{7}{12}} \rightarrow r = \frac{7}{15}h \).
   Then \( V = \frac{1}{3} \pi \left( \frac{7}{12}h \right)^2 h = 49\pi \frac{h^3}{432} \) and \( \frac{dV}{dh} = 49\pi \frac{h^2}{144} \).

4. (a) \( \frac{dP}{dt} = \frac{24}{\pi} \) in/sec.
   Hint: Want \( \frac{dP}{dt} \) and we know \( \frac{dC}{dt} = 6 \) in/sec. To relate \( P \) and \( C \) note that the side of the square is equal to \( 2r \) so \( P = 8r \). Then since \( C = 2\pi r \rightarrow r = \frac{C}{2\pi} \) So \( P = 8 \left( \frac{C}{2\pi} \right) = \frac{4C}{\pi} \) . It follows \( \frac{dP}{dC} = \frac{4}{\pi} \).
   (b) \( \frac{dA}{dt} = 120 \left( \frac{1}{\pi} - \frac{1}{4} \right) \) in\(^2\)/sec
   Hint: Want \( \frac{dA}{dt} \) and we know \( \frac{dC}{dt} = 6 \) in/sec. To relate \( A \) and \( C \) note that the side of the square is equal to \( 2r \) so \( A_{\square} = (2r)^2 \). Then since \( C = 2\pi r \rightarrow r = \frac{C}{2\pi} \) So \( A_{\square} = \left( 2 \cdot \frac{C}{2\pi} \right)^2 = \frac{C^2}{\pi^2} \). Then \( A = \frac{C^2}{\pi^2} - \frac{C^2}{4} \).
   It follows \( \frac{dA}{dC} = 2C \left( \frac{1}{\pi^2} - \frac{1}{4\pi} \right) \). When \( A = 25\pi \rightarrow r = 5 \) so \( C = 10\pi \).

5. \( \frac{10}{3} \) cm\(^2\)/sec.
   Hint: Want \( \frac{dA}{dt} \) and we know \( \frac{dV}{dt} = 10 \text{ cc/sec.} \). To relate \( A \) and \( V \) note that both are given in terms of \( r \):
   \( A = 4\pi r^2 \) and \( V = \frac{4}{3} \pi r^3 \) so \( V = \frac{4}{3} \pi \left( \frac{A}{4\pi} \right)^{3/2} \). It follows \( \frac{dV}{dA} = \frac{1}{2} \left( \frac{A}{4\pi} \right)^{1/2} \).

6. 800m of the $1/m$ fence and 200m of the $2/m$ fence. The cost will be $1200.
   Hint: If \( x \) is the front and \( y \) represents the sides, we have (1) \( x \cdot y = 60000 \) and (2) \( C = 2x + 1y + 1x + 1y \). Simplifying and substituting (1) we have \( C = 3x + 2 \left( \frac{60000}{x} \right) \).

7. \( r \approx 3.56 \text{ cm} \) and \( h \approx 8.91 \text{ cm} \).
   Hint: Want to minimize cost through surface area: \( C = (0.03)\pi r^2 + (0.02)2\pi rh + (0.02)\pi r^2 = (0.05)\pi r^2 + 0.04\pi rh. \)
   The additional constraint is \( V = \pi r^2 h = 355 \). Substituting the constraint for \( h \) gives \( C = (0.05)\pi r^2 + 0.04\pi r \left( \frac{355}{\pi r^2} \right) \).
   It follows we need to maximize \( C = (0.05)\pi r^2 + \frac{14.2}{r} \).
8. (a) When \( x = 1 \), \( A = \frac{1}{2} \).

Hint: Area of rectangle is given by base \((x)\) and height \((y = \frac{1}{x^2+1})\) so \( A = \frac{x}{x^2 + 1} \). Find max. of this function.

(b) No. IP at \( x = \sqrt{\frac{1}{3}} \)

Hint: Want \( y'' \) (NOT \( A'' \)). Note \( y'' = \frac{2(3x^2 - 1)}{(x^2 + 1)^3} \).

9. \( r = \frac{20\sqrt{3}}{3} \text{ cm}, \ h = \frac{40}{5} \text{ cm} \).

Hint: If \( r \) is the radius of the base of the cone, then using the radius of the sphere = 10cm and Pythagoras, we have the height of the cone, \( h = 10 + \sqrt{10^2 - r^2} \), assuming the cone extends below the equator.

Note: If \( h < 10 \), (so the base of the cone is above the equator) then we have \( h = 10 - \sqrt{10^2 - r^2} \). Can you see why there is a cone with the same base area but \( h > 10 \) so we can ignore all cones where \( h < 10 \)?

From this it follows \( V = \frac{1}{3} \pi r^2 (10 + \sqrt{10^2 - r^2}) \) BUT this produces a really unpleasant derivative.

Consider instead writing \( V \) as a function of \( h \). From above you should see that \( r^2 = 20h - h^2 \). Use this to help.

10. We want to maximize \( A = xy \) with the constraint that \( 2x + 2y = k \). \( 2x + 2y = k \rightarrow y = \frac{k - 2x}{2} \).

Then \( A = x \left( \frac{k - 2x}{2} \right) = \frac{1}{2} kx - x^2 \). So \( \frac{dA}{dx} = \frac{k}{2} - 2x = 0 \rightarrow x = \frac{k}{4} \). Since \( \frac{d^2A}{dx^2} = -2 \), this is a maximum.

It follows that since \( y = \frac{k - 2(\frac{k}{4})}{2} = \frac{k}{4} \), the rectangle is a square.

11. (a) 3573.3’ from the box along the road.

Hint: Want to minimize cost: \( C = 35\sqrt{900^2 + x^2} + 20(4200 - x) \). Notice the domain bounds: \( [0, 4200] \) and what they mean in this context. \( x = 4200 \) would be the shortest overall distance, while \( x = 0 \) would be the shortest path through the forest.

(b) \$109,850.53 (Note that \( C(0) = \$115,500 \))

(c) \$150,337.12 (when \( x = 4200 \))

12. \(36\pi\)

13. \(\frac{\pi}{4}\)

14. (a) 20’ for circle and 0’ for square.

(b) \(\frac{20\pi}{\pi+4}\)’ for circle and \(\frac{80}{\pi+4}\)’ for square.

15. \$37