The Fundamental Theorem of Calculus for Line Integrals

We’ve seen two ways of evaluating line integrals,

\[
\Diamond \text{ By parameterization: } \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt
\]

\[
\Diamond \text{ By the Fundamental Theorem of Line Integrals: } \int_C \vec{F} \cdot d\vec{r} = \int_a^b \nabla f \cdot d\vec{r} = f(b) - f(a)
\]

Where the condition on the FTLI is that there exists \( f \) such that \( \nabla f = \vec{F} \).

This is guaranteed (on a parallelogram) so long as \( \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \) where \( \vec{F} = F_1 \vec{i} + F_2 \vec{j} \).
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The relationship between \( \frac{\partial F_1}{\partial y} \) and \( \frac{\partial F_2}{\partial x} \) is common enough in vector calculus to receive it’s own definition:

\[ \text{curl}(\vec{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \]

It follows that a gradient field has a curl of 0.
We have seen that for gradient fields the line integral \( \int_C \vec{F} \cdot d\vec{r} \) about a closed path is 0.

The study of line integrals about closed paths in general gave rise to the theorem we attribute to George Green (though Gauss seems to have beaten him to it).

The theorem requires (at least initially) a simple closed curve, \( C \), that is the boundary of some elementary region, \( R \). (The boundary of a region, \( R \), is often denoted \( \partial R \).)
Green’s Theorem

If \( \vec{F}(x, y) = F_1(x, y)\vec{i} + F_2(x, y)\vec{j} \) is continuously differentiable and \( R \) is a simple closed region with boundary \( C \) traversed counterclockwise (so the interior is always on the left),
Then

\[
\int \int _{ R } \left( \frac{ \partial F_2 }{ \partial x } - \frac{ \partial F_1 }{ \partial y } \right) \, dx \, dy = \oint _{ C } F_1 \, dx + F_2 \, dy
\]

The closed path, \( C \), is indicated by the integral symbol \( \oint \).
Note the integrand on the left is given by \( \text{curl}(\vec{F}) \).

Figure 5: Simple closed region \( R \) with boundary \( C \).
Example: Use Green’s theorem to compute the work done by the vector field $\vec{F}(x, y) = (2x^2 + y)i + (3x)j$ in moving a particle once around the perimeter of the rectangle shown in the figure in the counterclockwise direction.
Example: Use Green’s theorem to compute the work done by the vector field $\vec{F}(x, y) = (2x^2 + y)\hat{i} + (3x)\hat{j}$ in moving a particle once around the perimeter of the rectangle shown in the figure in the counterclockwise direction.

Solution: The work is given by the line integral $\oint_C \vec{F} \cdot d\vec{r} = \oint_C (2x^2 + y)\,dx + (3x)\,dy$.

Then $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 3 - 1 = 2$ so Green’s theorem gives us

$$\iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \,dx\,dy = \iint_R 2\,dx\,dy = 2 \iint_R \,dx\,dy$$

where $R$ is the area of the rectangle.

The area of the rectangle is 8 so the integral (and therefore the work done by $\vec{F}$) is 16.
**Example:** Use Green’s theorem to compute the work done by the vector field
\[ \vec{F}(x, y) = (y + 3x) \vec{i} + (2y - x) \vec{j} \]
in moving a particle once around the ellipse \( 4x^2 + y^2 = 4 \) in the counterclockwise direction.
Example: Use Green’s theorem to compute the work done by the vector field \( \vec{F}(x, y) = (y + 3x)i + (2y - x)j \) in moving a particle once around the ellipse \( 4x^2 + y^2 = 4 \) in the counterclockwise direction.

Solution: The work is given by the line integral
\[
\oint_C \vec{F} \cdot d\vec{r} = \oint_C (y + 3x)\,dx + (2y - x)\,dy.
\]

Figure 8: Elliptical path through \( \vec{F} = (y + 3x)i + (2y - x)j \)
**Example:** Use Green’s theorem to compute the work done by the vector field \( \vec{F}(x, y) = (y + 3x)\vec{i} + (2y - x)\vec{j} \) in moving a particle once around the ellipse \( 4x^2 + y^2 = 4 \) in the counterclockwise direction.

**Solution:** The work is given by the line integral

\[
\oint_C \vec{F} \cdot d\vec{r} = \oint_C (y + 3x) \, dx + (2y - x) \, dy.
\]

Since \( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -1 - 1 = -2 \), Green’s theorem gives us

\[
\iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \, dy = \iint_R -2 \, dx \, dy
\]

where \( R \) is the area of the ellipse.

Since the axes of the ellipse are 1 and 2 respectively, the area of the ellipse is \( \pi(1)(2) = 2\pi \) and the work is \(-4\pi\).

![Figure 9: Elliptical path through \( \vec{F} = (y + 3x)\vec{i} + (2y - x)\vec{j} \)](image)
Example: Evaluate $\oint_C (5 - xy - y^2) \, dx - (2xy - x^2) \, dy$ where $C$ is the square with vertices $(0, 0), (1, 0), (1, 1), (0, 1)$. 
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Solution: From Green’s theorem we have \( \oint_C (5 - xy - y^2) \, dx - (2xy - x^2) \, dy = \iint_R 3x \, dx \, dy \)

Which gives us \( \int_0^1 \int_0^1 3x \, dx \, dy = \int_0^1 \left[ \frac{3}{2} x^2 \right]_0^1 dy = \frac{3}{2} y \bigg|_0^1 = \frac{3}{2} \)
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Note the alternative is to evaluate the line integral by parameterization:

$\oint_C (5 - xy - y^2) \, dx - (2xy - x^2) \, dy$ with

$$\vec{r}'(t) = \begin{cases} \vec{t} & : 0 \leq t \leq 1 \\ \vec{i} + (t - 1)\vec{j} & : 1 \leq t \leq 2 \\ (3 - t)\vec{i} + \vec{j} & : 2 \leq t \leq 3 \\ (4 - t)\vec{j} & : 3 \leq t \leq 4 \end{cases}$$

Which is manageable but tedious.
One of the consequences of Green’s theorem involves the area of $R$.

**Area of a Region**

If $C$ is the simple closed path that bounds the region $R$ to which Green’s theorem applies, then the area of the region is given by

$$A = \frac{1}{2} \oint_C x \, dy - y \, dx$$
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**Proof:** Note that $\oint_C x \, dy - y \, dx = \oint_C -y \, dx + x \, dy$ so when we apply Green's theorem we have

$$\frac{1}{2} \oint_C x \, dy - y \, dx = \frac{1}{2} \oint_C -y \, dx + x \, dy$$

$$= \frac{1}{2} \iint_R (1 - (-1)) \, dx \, dy$$

$$= \iint_R dx \, dy$$

$$= A$$
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**Area of a Region**

If \( C \) is the simple closed path that bounds the region \( R \) to which Green’s theorem applies, then the area of the region is given by

\[
A = \frac{1}{2} \oint_C x \, dy - y \, dx
\]

There are several vector fields with this property. Can you name some others?
Example: Find the area of the hypocycloid described by $\vec{r}(t) = (\cos^3 t)\vec{i} + (\sin^3 t)\vec{j}$ for $0 \leq t \leq 2\pi$. 
Example: Find the area of the hypocycloid described by $\vec{r}(t) = (\cos^3 t)i + (\sin^3 t)j$ for $0 \leq t \leq 2\pi$.

Solution: From the previous theorem we know $A = \frac{1}{2} \oint_C x \, dy - y \, dx$ so using the parameterization we have $x = \cos^3 t$ and $y = \sin^3 t$ so $\frac{dx}{dt} = -3 \cos^2 t \sin t$ and $\frac{dy}{dt} = 3 \sin^2 t \cos t$: 

![Hypocycloid Diagram](image-url)
\[
\frac{1}{2} \oint_C x \, dy - y \, dx = \frac{1}{2} \int_0^{2\pi} 3 \cos^4 t \sin^2 t + 3 \cos^2 t \sin^4 t \, dt \\
= \frac{3}{2} \int_0^{2\pi} \cos^2 t \sin^2 t (\cos^2 t + \sin^2 t) \, dt \\
= \frac{3}{2} \int_0^{2\pi} \cos^2 t \sin^2 t \, dt \quad \text{Now recall } \sin 2t = 2 \sin t \cos t \\
= \frac{3}{2} \int_0^{2\pi} \frac{1}{4} \sin^2 (2t) \, dt \quad \text{and since } \sin^2 t = (1 - \cos t)/2 \\
= \frac{3}{8} \int_0^{2\pi} \frac{1 - \cos 4t}{2} \, dt \\
= \frac{3}{8} \pi
\]
Vector Form of Green’s theorem

\[ \oint_C \vec{F} \cdot d\vec{r} = \iint_R (\text{curl} \vec{F}) \cdot \vec{k} \, dA = \iint_R (\nabla \times \vec{F}) \cdot \vec{k} \, dA \]
Vector Form of Green’s theorem

\[ \oint_C \vec{F} \cdot d\vec{r} = \iint_R (\text{curl}\, \vec{F}) \cdot \hat{k} \, dA = \iint_R (\nabla \times \vec{F}) \cdot \hat{k} \, dA \]

**Example:** Let \( \vec{F} = (xy^2)i + (y + x)j \). Integrate \((\nabla \times \vec{F}) \cdot \hat{k}\) over the region in the first quadrant bounded by \( y = x^2 \) and \( y = x \).
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**Example:** Let \( \vec{F} = (xy^2)\hat{i} + (y + x)\hat{j} \). Integrate \( (\nabla \times \vec{F}) \cdot \vec{k} \) over the region in the first quadrant bounded by \( y = x^2 \) and \( y = x \).

**Solution:** (one way)

\[ \nabla \times \vec{F} = 0\hat{i} + 0\hat{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k} = (1 - 2xy)\hat{k} \]

Then \( (\nabla \times \vec{F}) \cdot \vec{k} = 1 - 2xy \)

So

\[ \iint_R (\nabla \times \vec{F}) \cdot \vec{k} \, dA = \int_0^1 \int_{x^2}^x (1 - 2xy) \, dy \, dx \]

\[ = \int_0^1 [y - xy^2]_{x^2}^x \, dx \]

\[ = xy - \frac{1}{2}x^2y^2 \bigg|_0^1 \]

\[ = \frac{1}{2} \]
**Conclusion:** Evaluating line integrals boils down to a decision process beginning with the curl test. For Line integrals in the plane this is described by the flow chart below: