What is Linear Algebra anyway?
Linear algebra is the branch of mathematics concerned with the study of

Vectors and matrices

Vector spaces (or linear spaces)

Linear transformations

Systems of linear equations
Linear algebra has extensive applications in

The natural sciences
  Physics
  Biology
  Chemistry

The social sciences

Engineering and Computer Science
A linear equation in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

where the coefficients $a_1, a_2, \ldots, a_n$ and $b$ are real or complex numbers which are generally known in advance.

The subscript $n$ may be any positive integer.
A system of linear equations is a collection of one or more linear equations which involve the same variables, call them $x_1, \ldots, x_n$. A typical 3x3 linear system looks like

\[
\begin{align*}
4x_1 + 7x_2 - 8x_3 &= 11 \\
2x_1 + 3x_2 - x_3 &= 0 \\
-3x_1 - 9x_2 + 8x_3 &= -1
\end{align*}
\]
Definition

A *solution* of the system is a list \((s_1, s_2, s_3)\) of numbers which makes each equation a true statement when the values \(s_1, s_2, s_3\) are substituted for \(x_1, x_2, x_3\) respectively.
Definition

The set of all possible solutions is called the solution set of the linear system.

Definition

Two linear systems are said to be equivalent if they have the same solution set.
What comes next ...

Linear equations will lead to the geometry of planes in space. Intuition is easy for small systems ... linear algebra will allow us to consider larger systems where intuition breaks down.

We introduce matrix notation to make our work easier.

We examine the (somewhat rare) cases when Gaussian Elimination runs into difficulties.

We estimate the number of elimination steps to solve a system of size $n$. 

For our first example, \( x + y = 4 \), \( x - y = -2 \), it is easy to visualize the geometry and the solution.
What about $x + y = 4$, $x + y = 2$? No solution!
What happens in this case?

\[ x + y = 4, \quad x + y = 2 \]

To solve this system by *elimination*, subtract the first equation from the second leaving one "equation", \( 0 = -2 \).

Definition
This is a *singular case* (no solution).
What about $x + y = 4$, $2x + 2y = 8$? Many solutions!
What does Gaussian elimination say in this case?

\[ x + y = 4, \quad 2x + 2y = 8 \]

To solve this system by elimination, subtract twice the first equation from the second leaving one equation.

\[ 0 = 0 \Rightarrow \text{this is a singular case with infinitely many solutions; } y \text{ can have any value.} \]
\[ xy + 6uv - \pi zw = 8t - 22 \]

Which are true?

(a) If the variables are \( y, v, w, \) and \( t \) (and \( x, u, \pi, \) and \( z \) are constants) then the equation is linear.

(b) If the variables are just \( x \) and \( z \), and all other symbols are constants, then the equation is linear.

(c) If \( x \) and \( y \) are variables then the equation is not linear.
What do you think?

(a) It looks 3-dimensional.
(b) It shows 3 planes intersecting in 3 lines.
(c) It shows that 3 planes do not have to have any points in their intersection.
(d) Two planes certainly intersect in a line.
Solving a system

\[ ax_1 + bx_2 = 5 \]
\[ cx_1 + dx_2 = 6 \]

suppose we find that

\[ x_1 = 3 \]
\[ x_2 = 5 \]

Is that one solution or two?
Consider the simple arithmetic problem

\[ 5x = 3. \]

If you multiply the equation by 8, do you get the same answer to the new equation?

Consider the equation and some solutions

\[ x - 2y = 0, \quad (x, y) = (0, 0), \ (2, 1), \ (4, 2). \]

(a) If you multiply the equation by 8, do you get the same answers to the new equation?

(b) If you multiply the answers by 8, do you get more answers to the original equation?
Consider the system of equations

\[ \begin{align*}
  x_1 &= 0 \\
  x_2 &= 0 \\
  0 &= 0.
\end{align*} \]

Which statements apply?

(a) There is one answer.
(b) There is no requirement about \( x_3 \).
(c) There are infinitely many solutions.
Breather
Matrix notation. We want a more convenient, compact method for writing a linear system. To this end we will use a rectangular array called a matrix.

Suppose we have the linear system

\[
\begin{align*}
x & + 2y + 3z = 6 \\
2x & + 5y + 2z = 4 \\
6x & - 3y + z = 2
\end{align*}
\]

The coefficients of each variable are aligned in columns and we can write the coefficient matrix as

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 5 & 2 \\
6 & -3 & 1
\end{bmatrix}
\]
In order to incorporate the right hand side values in the system we can write the *augmented matrix* of the system. We append the right hand side column to the coefficient matrix to get:

\[
\begin{bmatrix}
1 & 2 & 3 & 6 \\
2 & 5 & 2 & 4 \\
6 & -3 & 1 & 2
\end{bmatrix}
\]
The size of a matrix gives the number of rows and columns.

Our coefficient matrix has 3 rows and 3 columns and is called a "3 by 3" \((3 \times 3)\) matrix. The augmented matrix has 3 rows and 4 columns and is called a "3 by 4" \((3 \times 4)\) matrix.

(The number of rows always comes first!!)
To find a solution for the system, we now must look for numbers $x, y, z$ that solve all three equations at once.

Those numbers may not exist - in this case they do.

When the number of equations matches the number of unknowns, there is frequently only one solution.
How can we visualize the problem?
There are two ways:

1. In terms of rows, each equation represents a plane. A solution is the point at which the three planes meet.

2. In terms of columns, solving the system amounts to combining three columns to produce the column vector \[
\begin{bmatrix}
6 \\
4 \\
2
\end{bmatrix}.
\]
In the row picture, each equation is a *plane* in 3-space.

The first plane comes from the equation $x + 2y + 3z = 6$.

This plane crosses the $x$, $y$, and $z$ axes at the points $(6,0,0)$, $(0,3,0)$, and $(0,0,2)$.

Those three points solve the equation and they determine the whole plane.
The vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ does not solve $x + 2y + 3z = 6$.

The plane defined by $x + 2y + 3z = 6$ does not pass through the origin.

Note that the plane $x + 2y + 3z = 0$ does pass through the origin and it is parallel to the plane $x + 2y + 3z = 6$. 
The second plane comes from the equation $2x + 5y + 2z = 4$.

This plane intersects the first plane in a line $L$.

The usual result of two equations in three unknowns is a line, $L$, of solutions.
The third equation defines a third plane.

It cuts the line $L$ at a single point.

That point lies on all three planes and it solves all three equations.

Now for the column picture ...
The column picture begins with the *vector form* of the equations.

\[
\begin{bmatrix}
1 \\
2 \\
6
\end{bmatrix}
\cdot
x + \begin{bmatrix}
2 \\
5 \\
-3
\end{bmatrix}
\cdot
y + \begin{bmatrix}
3 \\
2 \\
1
\end{bmatrix}
\cdot
z = \begin{bmatrix}
6 \\
4 \\
2
\end{bmatrix}
\]

The unknowns \( x, y, z \) are the coefficients in this *linear combination*. 
The unknowns $x, y, z$ are called the coefficients of this linear combination. We want to multiply the three column vectors on the left by the correct numbers $x, y, z$ to produce the right hand side column vector $\vec{b} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$. 

\[
\begin{bmatrix} x \\ 2 \\ 6 \end{bmatrix} + y \cdot \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}
\]
So ...

Row picture ... Intersection of planes

Column picture ... Combination of columns
For the system

\[
x \cdot \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y \cdot \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}
\]

The triplet \((x, y, z) = (0, 0, 2)\) is the solution since

\[
0 \cdot \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + 2 \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \vec{b}.
\]
What about the matrix form of the equations?
We have the linear system

\[
\begin{align*}
    x & + \ 2y & + \ 3z & = \ 6 \\
    2x & + \ 5y & + \ 2z & = \ 4 \\
    6x & - \ 3y & + \ z & = \ 2
\end{align*}
\]

The coefficient matrix is

\[
\begin{bmatrix}
    1 & 2 & 3 \\
    2 & 5 & 2 \\
    6 & -3 & 1
\end{bmatrix}
\]

The right hand side vector is

\[
\begin{bmatrix}
    6 \\
    4 \\
    2
\end{bmatrix}
\]
The system can be written as a matrix equation:

\[
\begin{bmatrix}
  1 & 2 & 3 \\
  2 & 5 & 2 \\
  6 & -3 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}
= 
\begin{bmatrix}
  6 \\
  4 \\
  2
\end{bmatrix}
\]
The matrix equation

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 5 & 2 \\
6 & -3 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
6 \\
4 \\
2
\end{bmatrix}
\]

in shorthand form is \( A\vec{x} = \vec{b} \).

We multiply the coefficient matrix \( A \) times the unknown vector \( \vec{x} \) to get the right-hand side vector \( \vec{b} \).

The question is: what does it mean to multiply \( A \) times \( \vec{x} \)?
$A\tilde{x}$ is a combination of column vectors:

\[
A\tilde{x} = x \cdot (\text{column 1}) + y \cdot (\text{column 2}) + z \cdot (\text{column 3}).
\]

\[
A\tilde{x} = x \cdot \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y \cdot \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x + 2y + 3z \\ 2x + 5y + 2z \\ 6x - 3y + z \end{bmatrix}
\]
Back to the matrix equation

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 5 & 2 \\
6 & -3 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
6 \\
4 \\
2
\end{bmatrix}
\]

When we substitute the solution \( \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \),
the multiplication \( A \vec{x} \) produces \( \vec{b} \).

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 5 & 2 \\
6 & -3 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
2
\end{bmatrix} = 2 \times ( \text{column 3} ) =
\begin{bmatrix}
6 \\
4 \\
2
\end{bmatrix}.
\]
Breather
Our goal is to look beyond two or three dimensions into \( n \) dimensions.

With \( n \) equations in \( n \) unknowns, there are \( n \) planes in the row picture.

There are \( n \) vectors in the column picture, plus a vector \( \vec{b} \) on the right hand side.
The equations ask for a *linear combination of the n columns that equals* $\bar{b}$.

With some equations that will not be possible! When that happens we have what is called a *singular* case.

What can go wrong? Looking at the end view of three planes ...
two parallel planes  no intersection

Planes viewed edge-on
line of intersection    all planes parallel

Planes viewed edge-on
In the row picture the singular case occurs when the three planes have no point in common - there is no intersection point.

In the column picture the singular case occurs when the three columns lie in the same plane and so are only solvable for $\tilde{b}$ in that plane.
Solving a general linear system.

We want to develop a systematic procedure or algorithm for solving linear systems.

Basic strategy:

We replace one system with an equivalent system, having the same solution set, but that is easier to solve.
We have three basic operations to simplify a linear system:

1) Replace one equation by the sum of itself and a multiple of another equation.

2) Interchange two equations.

3) Multiply all terms in an equation by a nonzero constant.

Any combination of these operations does not change the solution set of the original linear system.
What does "easier to solve" mean? What is the goal of the algorithm?

To change the original augmented linear system matrix

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\
a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\
a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\
a_{41} & a_{42} & a_{43} & a_{44} & b_4 \\
\end{bmatrix}
\]

into the form

\[
\begin{bmatrix}
\square & * & * & * & \star \\
0 & \square & * & * & \star \\
0 & 0 & \square & * & \star \\
0 & 0 & 0 & \square & \star \\
\end{bmatrix}
\]

Where the leading entries (\(\square\)) may have any nonzero value; the starred entries (\(\star\)) may have any values, including zero.
Row Reduction and Echelon forms.

**Definition** A rectangular matrix is in *row echelon form* if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zeros.
If, in addition:

1. The leading entry in each nonzero row is 1, and

2. Each leading 1 is the only nonzero entry in its column, then the matrix is said to be in **reduced row echelon** form.
Definition
Two matrices are said to be *row equivalent* if there is a sequence of elementary row operations that transforms one matrix into the other.

Elementary row operations are:
1) Replacement
2) Interchange
3) Scaling
Theorem  Uniqueness of the Reduced Echelon Form. Each matrix is row equivalent to one and only one reduced echelon matrix.

Echelon form:

$$\begin{bmatrix}
\Box & * & * & * \\
0 & \Box & * & * \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

Reduced Echelon form:

$$\begin{bmatrix}
1 & 0 & * & * \\
0 & 1 & * & * \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$
Definition

A *pivot position* in a matrix $A$ is a location in $A$ that corresponds to a leading 1 in the reduced echelon form of $A$.

A *pivot column* is a column of $A$ that contains a pivot position.
Theorem - Existence and Uniqueness

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column.

This means that an echelon form of the augmented matrix has *no* rows of the form

\[
\begin{bmatrix}
0 & \ldots & 0 & \mid & b
\end{bmatrix}
\]

with \( b \) nonzero
If a linear system is consistent, then the solution set contains either a unique solution, where there are no free variables, or infinitely many solutions, when there is at least one free variable.
\[
\begin{pmatrix}
1 & 7 & 0 & 3 & 5 \\
1 & 0 & 1 & 0 & 16 \\
2 & 14 & 0 & 0 & 10 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 7 & 0 & 3 & 5 \\
2 & 0 & 2 & 0 & 32 \\
0 & 0 & 0 & -6 & 0 \\
\end{pmatrix}
\]

Which row operation was not done?

a) (row 2) \rightarrow 2(row 2)

b) interchange (row 2), (row 3)

c) (row 3) \rightarrow (row 3) - 2(row 1)
Suppose a matrix $A$ is transformed to matrix $B$ by the row operation:
(row 2) $\rightarrow$ (row 2) $- 3$(row 5)
Which operation transforms $B$ back to $A$?
(a) (row 2) $\rightarrow$ (row 2) $- \frac{1}{3}$(row 5)
(b) (row 2) $\rightarrow$ (row 2) $+ 3$(row 5)
(c) You can’t do it.
Which of these matrices are in reduced row-echelon form?

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3 \\
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 13 & 5 \\
0 & 1 & 0 & 16 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
How about the number of leading 1’s in the reduced row-echelon form of a $3 \times 4$ matrix?

(a) There must be 3 of them.
(b) There may be 0, 1, 2, or 3.
(c) There could be 4.
Using that

\[ 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 \\ -17 \end{bmatrix} \]

(or not), find a solution to

\[ \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix} \vec{x} = \begin{bmatrix} -4 \\ -17 \end{bmatrix} \]
Suppose we found two solutions to the same equation: $Ax = b$, and $Ay = b$.
True–False: we can get another solution to the same equation by just adding: $x + y$ is also a solution.
Suppose that $A\vec{u} = \vec{b}$ and $A\vec{z} = \vec{0}$. Which are true?

(a) $A(\vec{u} + \vec{z}) = \vec{b}$

(b) $A(\vec{u} - \vec{v}) = \vec{0}$ and therefore $\vec{u} - \vec{v} = \vec{z}$

(c) $A(\vec{u} - \vec{v}) = \vec{0}$
Suppose we know solutions of the two equations: $A\vec{x} = \vec{b}$, and $A\vec{y} = \vec{0}$. Which is correct?

(a) We can get another solution of the first equation by just adding: $\vec{x} + \vec{y}$ is also a solution.

(b) $\vec{x} + \vec{y}$ is a solution of the second equation.
End Presentation