Chapter 4

Exponential Functions

"There is not much difference between the delight a novice experiences in cracking a clever brain teaser and the delight a mathematician experiences in mastering a more advanced problem. Both look on beauty bare—that clean, sharply defined, mysterious, entrancing order that underlies all structure."

—*Martin Gardner*

Table 1 Numbers of Starbucks Stores

Year	Number of Stores
1991	116
1993	272
1995	676
1997	1412
1999	2135
2001	4709
2003	7225

Source: *Starbucks Corporation*

Is there a Starbucks® store nearby? The number of these coffeehouses worldwide has increased greatly since 1991 (see Table 1). In Exercise 33 of Homework 4.5, you will predict the number of stores in 2010 and describe the increase in the number of stores over time.

In Chapters 1–3, we worked with linear expressions and equations. In this chapter, we will work with a new type of expression and equation. In addition to working with linear functions, we will now use *exponential functions* to model some authentic situations. We will use these functions to make predictions, such as predicting the number of Starbucks stores in 2010.

4.1 PROPERTIES OF EXPONENTS

Objectives

- Know the meaning of *exponent,* zero exponent, and negative exponent.
- Know properties of exponents.
- Simplify expressions involving exponents.
- Know the meaning of *exponential function.*
- Use *scientific notation.*

In this section, we will simplify expressions involving *exponents*.

Definition of an Exponent

If n is a counting number (Section A.2), what is the meaning of b^n? The notation b^3 stands for $b \cdot b \cdot b$. So, $4^3 = 4 \cdot 4 \cdot 4 = 64$. The notation b^4 stands for $b \cdot b \cdot b \cdot b$. So, $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$.

DEFINITION Exponent

For any counting number n,

$$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ factors of } b}$$

We refer to b^n as the **power,** the ***n*th power of *b*,** or ***b* raised to the *n*th power.** We call b the **base** and n the **exponent.**

The expression 3^5 is a power. It is the 5th power of 3, or 3 raised to the 5th power. For 3^5, the base is 3 and the exponent is 5. Here, we label the base and the exponent of 3^5 and calculate the power:

$$\overset{\text{Exponent}}{3^5} = \underbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}_{5 \text{ factors of } 3} = 243$$

Base

When we calculate a power, we say that we have performed an **exponentiation.**

Notice that the notation b^1 stands for one factor of b, so $b^1 = b$.

Two powers of b have specific names. We refer to b^2 as the **square of *b*** or ***b* squared.** We refer to b^3 as the **cube of *b*** or ***b* cubed.**

For an expression of the form $-b^n$, we compute b^n before finding the opposite. For example,

$$-2^4 = -(2^4) = -(2 \cdot 2 \cdot 2 \cdot 2) = -16$$

For -2^4, the base is 2, *not* -2. If we want the base to be -2, we must enclose -2 in parentheses:

$$(-2)^4 = (-2)(-2)(-2)(-2) = 16$$

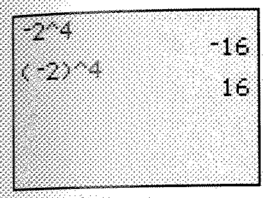

Figure 1 Compute -2^4 and $(-2)^4$

We can use a graphing calculator to check both computations (see Fig. 1). To find -2^4, press [(−)] **2** [∧] **4** [ENTER].

Properties of Exponents

In this section and Section 4.2, we discuss five properties of exponents.

Properties of Exponents

If m and n are counting numbers, then

- $b^m b^n = b^{m+n}$ — *Product property for exponents*
- $\dfrac{b^m}{b^n} = b^{m-n}$, $b \neq 0$ and $m > n$ — *Quotient property for exponents*
- $(bc)^n = b^n c^n$ — *Raising a product to a power*
- $\left(\dfrac{b}{c}\right)^n = \dfrac{b^n}{c^n}$, $c \neq 0$ — *Raising a quotient to a power*
- $(b^m)^n = b^{mn}$ — *Raising a power to a power*

In Example 1 (and the first Exploration), we will investigate why these properties make sense.

Example 1 Meaning of Exponential Properties

1. Show that $b^2b^3 = b^5$.
2. Show that $b^m b^n = b^{m+n}$, where m and n are counting numbers.
3. Show that $\left(\dfrac{b}{c}\right)^n = \dfrac{b^n}{c^n}$, where n is a counting number and $c \neq 0$.

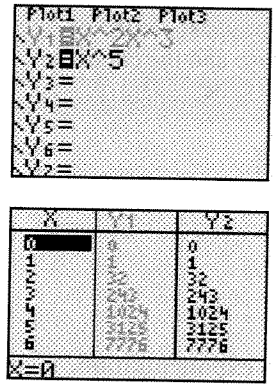

Figure 2 Comparing tables for $y = x^2x^3$ and $y = x^5$

Solution

1. By writing b^2b^3 without exponents, we see that

$$\begin{aligned} b^2b^3 &= (bb)(bbb) && \text{Write without exponents.} \\ &= bbbbb && \text{Remove parentheses.} \\ &= b^5 && \text{Write with an exponent.} \end{aligned}$$

We can verify that this result is correct for various constant bases by examining graphing calculator tables for both $y = x^2x^3$ and $y = x^5$ (see Fig. 2). For graphing calculator instructions, see Section B.14.

2. We write $b^m b^n$ without exponents:

$$b^m b^n = (\underbrace{bbb\cdots b}_{m \text{ factors}})(\underbrace{bbb\cdots b}_{n \text{ factors}}) = \underbrace{bbb\cdots b}_{m+n \text{ factors}} = b^{m+n}$$

3. We write $\left(\dfrac{b}{c}\right)^n$, where $c \neq 0$, without exponents:

$$\left(\frac{b}{c}\right)^n = \underbrace{\left(\frac{b}{c}\right)\left(\frac{b}{c}\right)\left(\frac{b}{c}\right)\cdots\left(\frac{b}{c}\right)}_{n \text{ factors}} = \frac{\overbrace{bbb\cdots b}^{n \text{ factors}}}{\underbrace{ccc\cdots c}_{n \text{ factors}}} = \frac{b^n}{c^n}$$ ■

Simplifying Expressions Involving Exponents

We can use properties of exponents to simplify expressions involving exponents.

> **Simplifying Expressions Involving Exponents**
>
> An expression involving exponents is simplified if
>
> 1. It includes no parentheses.
> 2. Each variable or constant appears as a base as few times as possible. For example, we write $x^2x^4 = x^6$.
> 3. Each numerical expression (such as 7^2) has been calculated, and each numerical fraction has been simplified.
> 4. Each exponent is positive.

Example 2 Simplifying Expressions Involving Exponents

Simplify.

1. $(2b^2c^3)^5$
2. $(3b^3c^4)(2b^6c^2)$
3. $\dfrac{3b^7c^6}{12b^2c^5}$
4. $\left(\dfrac{24b^7c^8}{16b^2c^5d^3}\right)^4$

Solution

1. $$\begin{aligned} (2b^2c^3)^5 &= 2^5(b^2)^5(c^3)^5 && \text{Raise factors to } n\text{th power: } (bc)^n = b^nc^n \\ &= 32b^{10}c^{15} && \text{Multiply exponents: } (b^m)^n = b^{mn} \end{aligned}$$

2. $$\begin{aligned} (3b^3c^4)(2b^6c^2) &= (3\cdot 2)(b^3b^6)(c^4c^2) && \text{Rearrange factors.} \\ &= 6b^9c^6 && \text{Add exponents: } b^mb^n = b^{m+n} \end{aligned}$$

3. $$\begin{aligned} \frac{3b^7c^6}{12b^2c^5} &= \frac{b^{7-2}c^{6-5}}{4} && \text{Subtract exponents: } \frac{b^m}{b^n} = b^{m-n} \\ &= \frac{b^5c}{4} && \text{Subtract.} \end{aligned}$$

4. $$\begin{aligned} \left(\frac{24b^7c^8}{16b^2c^5d^3}\right)^4 &= \left(\frac{3b^5c^3}{2d^3}\right)^4 && \text{Subtract exponents: } \frac{b^m}{b^n} = b^{m-n} \\ &= \frac{(3b^5c^3)^4}{(2d^3)^4} && \text{Raise numerator and denominator to } n\text{th power: } \left(\frac{b}{c}\right)^n = \frac{b^n}{c^n} \\ &= \frac{3^4(b^5)^4(c^3)^4}{2^4(d^3)^4} && \text{Raise factors to } n\text{th power: } (bc)^n = b^nc^n \\ &= \frac{81b^{20}c^{12}}{16d^{12}} && \text{Multiply exponents: } (b^m)^n = b^{mn} \end{aligned}$$ ■

WARNING The expressions $3b^2$ and $(3b)^2$ are *not* equivalent expressions:

$$3b^2 = 3b \cdot b$$
$$(3b)^2 = (3b)(3b) = 9b \cdot b$$

For $3b^2$, the base is the variable b. For $(3b)^2$, the base is the product $3b$.

Here, we show a typical error and the correct way to find the power $(3b)^2$:

$$\begin{aligned} (3b)^2 &= 3b^2 && \text{Incorrect} \\ (3b)^2 &= 3^2b^2 = 9b^2 && \text{Correct} \end{aligned}$$

Since the base $3b$ is a product, we need to distribute the exponent 2 to *both* factors 3 and b.

In general, when finding a power of the form $(bc)^n$, don't forget to distribute the exponent n to both factors b and c.

Zero as an Exponent

What is the meaning of b^0? If the property $\dfrac{b^m}{b^n} = b^{m-n}$ is to be true for $m = n$, then

$$1 = \frac{b^n}{b^n} = b^{n-n} = b^0, \quad b \neq 0$$

So, a reasonable definition of b^0 is 1.

> **DEFINITION** Zero exponent
>
> For $b \neq 0$,
>
> $$b^0 = 1$$

For example, $7^0 = 1$, $(-3)^0 = 1$, and $(ab)^0 = 1$, where $ab \neq 0$.

Negative Exponents

If n is an integer (Section A.2), what is the meaning of b^n? In particular, what is the meaning of a negative integer exponent? If the property $\dfrac{b^m}{b^n} = b^{m-n}$ is to be true for $m = 0$, then

$$\frac{1}{b^n} = \frac{b^0}{b^n} = b^{0-n} = b^{-n}, \quad b \neq 0$$

So, we should define b^{-n} to be $\dfrac{1}{b^n}$.

DEFINITION Negative integer exponent

If $b \neq 0$ and n is a counting number, then

$$b^{-n} = \frac{1}{b^n}$$

In words: To find b^{-n}, take its reciprocal and switch the sign of the exponent.

For example, $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$ and $b^{-5} = \frac{1}{b^5}$.

Next, we write $\frac{1}{b^{-n}}$ in another form, where $b \neq 0$ and n is a counting number:

$$\begin{aligned} \frac{1}{b^{-n}} &= 1 \div b^{-n} && \frac{a}{b} = a \div b \\ &= 1 \div \frac{1}{b^n} && \text{Write power so exponent is positive: } b^{-n} = \frac{1}{b^n} \\ &= 1 \cdot \frac{b^n}{1} && \text{Multiply by reciprocal of } \frac{1}{b^n}\text{, which is } \frac{b^n}{1}. \\ &= b^n && \text{Simplify.} \end{aligned}$$

So, $\frac{1}{b^{-n}} = b^n$.

Negative Exponent in a Denominator

If $b \neq 0$ and n is a counting number, then

$$\frac{1}{b^{-n}} = b^n$$

In words: To find $\frac{1}{b^{-n}}$, take its reciprocal and switch the sign of the exponent.

For example, $\frac{1}{2^{-4}} = 2^4 = 16$ and $\frac{1}{b^{-8}} = b^8$.

Simplifying More Expressions Involving Exponents

As we have seen, simplifying an expression involving exponents includes writing the expression so that each exponent is positive.

Example 3 Simplifying Expressions Involving Exponents

Simplify.

1. $9b^{-7}$ **2.** $\frac{5}{b^{-3}}$ **3.** $3^{-1} + 4^{-1}$

Solution

1. $9b^{-7} = 9 \cdot \frac{1}{b^7} = \frac{9}{b^7}$

2. $\frac{5}{b^{-3}} = 5 \cdot \frac{1}{b^{-3}} = 5b^3$

3. $3^{-1} + 4^{-1} = \frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$ ■

It turns out that the five properties discussed at the start of this section are also true for all negative-integer exponents and the zero exponent.

Properties of Integer Exponents

If m and n are integers, $b \neq 0$, and $c \neq 0$, then

- $b^m b^n = b^{m+n}$ — Product property for exponents
- $\frac{b^m}{b^n} = b^{m-n}$ — Quotient property for exponents
- $(bc)^n = b^n c^n$ — Raising a product to a power
- $\left(\frac{b}{c}\right)^n = \frac{b^n}{c^n}$ — Raising a quotient to a power
- $(b^m)^n = b^{mn}$ — Raising a power to a power

Example 4 Simplifying Expressions Involving Exponents

Simplify.

1. $2^{-1003}2^{1000}$ **2.** $\frac{b^{-6}}{b^{-4}}$ **3.** $\frac{35b^{-9}c^3}{25b^{-7}c^{-5}}$

Solution

1.
$$\begin{aligned} 2^{-1003}2^{1000} &= 2^{-1003+1000} && \text{Add exponents: } b^m b^n = b^{m+n} \\ &= 2^{-3} && \text{Simplify.} \\ &= \frac{1}{2^3} && \text{Write powers so exponents are positive: } b^{-n} = \frac{1}{b^n} \\ &= \frac{1}{8} && \text{Simplify.} \end{aligned}$$

2.
$$\begin{aligned} \frac{b^{-6}}{b^{-4}} &= b^{-6-(-4)} && \text{Subtract exponents: } \frac{b^m}{b^n} = b^{m-n} \\ &= b^{-6+4} && a - b = a + (-b) \\ &= b^{-2} && \text{Simplify.} \\ &= \frac{1}{b^2} && \text{Write powers so exponents are positive: } b^{-n} = \frac{1}{b^n} \end{aligned}$$

3.
$$\begin{aligned} \frac{35b^{-9}c^3}{25b^{-7}c^{-5}} &= \frac{7b^{-9-(-7)}c^{3-(-5)}}{5} && \text{Subtract exponents: } \frac{b^m}{b^n} = b^{m-n} \\ &= \frac{7b^{-2}c^8}{5} && \text{Simplify.} \\ &= \frac{7c^8}{5b^2} && \text{Write powers so exponents are positive: } b^{-n} = \frac{1}{b^n} \end{aligned}$$
■

In the first step of Problem 2 of Example 4, we found that

$$\frac{b^{-6}}{b^{-4}} = b^{-6-(-4)}$$

WARNING We need a subtraction symbol *and* a negative symbol in the expression on the right-hand side. It is a common error to omit writing one of these two symbols in such problems.

Example 5 Simplifying Expressions Involving Exponents

Simplify.

1. $\dfrac{(3bc^5)^2}{(2b^{-2}c^2)^3}$ **2.** $\left(\dfrac{18b^{-4}c^7}{6b^{-3}c^2}\right)^{-4}$

Solution

1.
$$\frac{(3bc^5)^2}{(2b^{-2}c^2)^3} = \frac{3^2b^2(c^5)^2}{2^3(b^{-2})^3(c^2)^3}$$ Raise factors to a power: $(bc)^n = b^nc^n$
$$= \frac{9b^2c^{10}}{8b^{-6}c^6}$$ Multiply exponents: $(b^m)^n = b^{mn}$
$$= \frac{9b^{2-(-6)}c^{10-6}}{8}$$ Subtract exponents: $\frac{b^m}{b^n} = b^{m-n}$
$$= \frac{9b^8c^4}{8}$$ Simplify.

2.
$$\left(\frac{18b^{-4}c^7}{6b^{-3}c^2}\right)^{-4} = \left(3b^{-4-(-3)}c^{7-2}\right)^{-4}$$ Subtract exponents: $\frac{b^m}{b^n} = b^{m-n}$
$$= \left(3b^{-1}c^5\right)^{-4}$$ Simplify.
$$= 3^{-4}(b^{-1})^{-4}(c^5)^{-4}$$ Raise factors to nth power: $(bc)^n = b^nc^n$
$$= 3^{-4}b^4c^{-20}$$ Multiply exponents: $(b^m)^n = b^{mn}$
$$= \frac{b^4}{3^4c^{20}}$$ Write powers so exponents are positive: $b^{-n} = \frac{1}{b^n}$
$$= \frac{b^4}{81c^{20}}$$ Simplify. ■

Definition of an Exponential Function

In this chapter and Chapter 5, we will work with *exponential functions*. Here are some examples of such functions:

$$f(x) = 2(3)^x, \qquad g(x) = -7\left(\frac{1}{2}\right)^x, \qquad h(x) = 5^x$$

Notice that, in exponential functions, the variable appears as an exponent.

> **DEFINITION** Exponential function
>
> An **exponential function** is a function whose equation can be put into the form
> $$f(x) = ab^x$$
> where $a \neq 0$, $b > 0$, and $b \neq 1$. The constant b is called the **base.**

Example 6 Evaluating Exponential Functions

For $f(x) = 3(2)^x$ and $g(x) = 5^x$, find the following.

1. $f(3)$ **2.** $f(-4)$ **3.** $g(a+3)$ **4.** $g(2a)$

Solution

1. $f(3) = 3(2)^3 = 3 \cdot 8 = 24$

2. $f(-4) = 3(2)^{-4} = \dfrac{3}{2^4} = \dfrac{3}{16}$

3.
$$g(a+3) = 5^{a+3}$$ Substitute $a+3$ for x in 5^x.
$$= 5^a \cdot 5^3$$ Write as product: $b^{m+n} = b^mb^n$
$$= 125(5)^a$$ $5^3 = 125$; rearrange factors: $ab = ba$

4.
$$g(2a) = 5^{2a}$$ Substitute $2a$ for x in 5^x.
$$= (5^2)^a$$ $b^{mn} = (b^m)^n$
$$= 25^a$$ $5^2 = 25$ ■

WARNING It is a common error to confuse exponential functions such as $E(x) = 2^x$ with linear functions such as $L(x) = 2x$. For the *exponential* function $E(x) = 2^x$, the variable x is an *exponent*. For the *linear* function $L(x) = 2x^1$, the variable x is a *base*.

Scientific Notation

Now we will discuss how to use exponents to describe numbers in *scientific notation*. This will enable us to describe compactly a number whose absolute value is very large or very small. For example, Earth is approximately 4,500,000,000 years old. We write 4,500,000,000 in scientific notation:

$$4.5 \times 10^9$$

The symbol "×" stands for multiplication.

As another example, light can travel 1 mile in 0.00000537 second. We write 0.00000537 in scientific notation:

$$5.37 \times 10^{-6}$$

> **DEFINITION** Scientific notation
>
> A number is written in **scientific notation** if it has the form $N \times 10^k$, where k is an integer and either $-10 < N \leq -1$ or $1 \leq N < 10$.

Here are more examples of numbers in scientific notation:

$$5.2 \times 10^{17} \qquad 3.638 \times 10^9 \qquad -5.86 \times 10^{-12} \qquad 2.13 \times 10^{-84}$$

In Example 7, we will convert some numbers from scientific notation to standard decimal notation.

Example 7 Converting to Standard Decimal Notation

Simplify.

1. 5×10^3 **2.** 5×10^{-3}

Solution

1. $5 \times 10^3 = 5 \times 1000 = 5000$

We simplify $5 \times 10^3 = 5.0 \times 10^3$ by *multiplying* 5.0 by 10 three times, hence moving the decimal point three places to the *right:*

$$5.0 \times 10^3 = 5000.0 = 5000$$

three places to the right

2.
$$5 \times 10^{-3} = 5 \times \frac{1}{10^3}$$ Write powers so exponents are positive: $b^{-n} = \frac{1}{b^n}$
$$= \frac{5}{1} \times \frac{1}{1000}$$ $a = \frac{a}{1}$; simplify.
$$= \frac{5}{1000}$$ Multiply.
$$= 0.005$$ $\frac{5}{1000}$ is 5 thousandths.

We simplify 5.0×10^{-3} by *dividing* 5.0 by 10 three times, hence moving the decimal point three places to the *left:*

$$5.0 \times 10^{-3} = 0.005$$

three places to the left

The problems in Example 7 suggest the way to convert a number from scientific notation $N \times 10^k$ to standard decimal notation.

Converting from Scientific Notation to Standard Decimal Notation

To write the scientific notation $N \times 10^k$ in standard decimal notation, we move the decimal point of the number N as follows:

- If k is *positive,* we multiply N by 10 k times; hence, we move the decimal point k places to the *right.*
- If k is *negative,* we divide N by 10 k times; hence, we move the decimal k places to the *left.*

Example 8 Converting to Standard Decimal Notation

Write the number in standard decimal notation.

1. 3.462×10^5 **2.** 7.38×10^{-4}

Solution

1. We *multiply* 3.462 by 10 five times; hence, we move the decimal point of 3.462 five places to the *right:*

$$3.462 \times 10^5 = 346{,}200.0$$

five places to the right

2. We *divide* 7.38 by 10 four times; hence, we move the decimal point of 7.38 four places to the *left:*

$$7.38 \times 10^{-4} = 0.000738$$

four places to the left

In Examples 7 and 8, we converted numbers from scientific notation to standard decimal notation. In Example 9, we will investigate the way to convert numbers from standard decimal notation to scientific notation.

Example 9 Converting to Scientific Notation

Write the number in scientific notation.

1. 6,257,000,000 **2.** 0.00000721

Solution

1. In scientific notation, we would have

$$6.257 \times 10^k$$

We must move the decimal point of 6.257 nine places to the right to get 6,257,000,000. So, $k = 9$ and the scientific notation is

$$6.257 \times 10^9$$

2. In scientific notation, we would have

$$7.21 \times 10^k$$

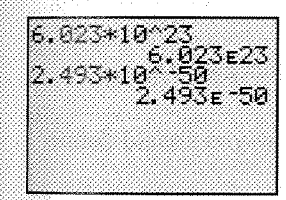

Figure 3 The numbers 6.023×10^{23} and 2.493×10^{-50}

We must move the decimal point of 7.21 six places to the left to get 0.00000721. So, $k = -6$ and the scientific notation is

$$7.21 \times 10^{-6}$$

The problems in Example 9 suggest the way to convert a number from standard decimal notation to scientific notation.

Converting from Standard Decimal Notation to Scientific Notation

To write a number in scientific notation, count the number of places k that the decimal point must be moved so that the new number N meets the condition $-10 < N \le -1$ or $1 \le N < 10$:

- If the decimal point is moved to the left, then the scientific notation is written as $N \times 10^k$.
- If the decimal point is moved to the right, then the scientific notation is written as $N \times 10^{-k}$.

Example 10 Converting to Scientific Notation

Write the number in scientific notation.

1. 92,900,000 (the average distance in miles between Earth and the Sun)
2. 0.0024 (the average weight, in grams, of a grain of sand)

Solution

1. For 92,900,000, the decimal point must be moved seven places to the left so that the new number is between 1 and 10. Therefore, the scientific notation is 9.29×10^7.
2. For 0.0024, the decimal point must be moved three places to the right so that the new number is between 1 and 10. Therefore, the scientific notation is 2.4×10^{-3}.

Calculators express numbers in scientific notation so that the numbers "fit" on the screen. To represent 6.023×10^{23}, most calculators use the notation 6.023 E 23, where E stands for exponent (of 10). Calculators represent 2.493×10^{-50} by 2.493 E −50 (see Fig. 3).

group exploration

Properties of exponents

1. In Example 1, we showed that the statement

$$b^2b^3 = b^5$$

makes sense by first writing the expression b^2b^3 without exponents. For each part, show that the given statement makes sense by first writing an expression without exponents.

a. $(bc)^4 = b^4c^4$

b. $\frac{b^7}{b^3} = b^4,\ b \ne 0$

c. $(b^3)^4 = b^{12}$

2. In Example 1, we also showed that the general statements

$$b^mb^n = b^{m+n} \quad \text{and} \quad \left(\frac{b}{c}\right)^n = \frac{b^n}{c^n},\ c \ne 0$$

make sense for counting numbers m and n. For each part, show that the general statement makes sense. Assume that m and n are counting numbers.

a. $(bc)^n = b^n c^n$

b. $\dfrac{b^m}{b^n} = b^{m-n}$, where $m > n$ and $b \neq 0$

c. $(b^m)^n = b^{mn}$

3. Choose values of b, c, and counting number n to show that the statement $(b+c)^n = b^n + c^n$ is false, in general.

group exploration

Looking ahead: Definition of $b^{1/n}$

Throughout this exploration, assume that $(b^m)^n = b^{mn}$ for rational numbers m and n.

1. First, you will explore the meaning of $b^{1/2}$, where b is nonnegative.

a. For now, do not use a calculator. You will explore how you should define $9^{1/2}$. You can determine a reasonable value of $9^{1/2}$ by first finding the *square* of the value:

$$\left(9^{1/2}\right)^2 = 9^{\frac{1}{2}\cdot 2} = 9^1 = 9$$

What would be a good meaning of $9^{1/2}$? **[Hint:** Can you think of a positive number whose square equals 9?]

b. What would be a good meaning of $16^{1/2}$? Of $25^{1/2}$?

c. Now use a graphing calculator to find $9^{1/2}$, $16^{1/2}$, and $25^{1/2}$. For example, to find $9^{1/2}$, press **9** [^] [(] **1** [÷] **2** [)] [ENTER]. Is the calculator interpreting $b^{1/2}$ as you would expect?

d. What would be a good meaning of $b^{1/2}$, where b is nonnegative?

2. Now you will explore the meaning of $b^{1/3}$.

a. For now, do not use a calculator. You will explore how you should define $8^{1/3}$. You can first find the *cube* of the value:

$$\left(8^{1/3}\right)^3 = 8^{\frac{1}{3}\cdot 3} = 8^1 = 8$$

What would be a good meaning of $8^{1/3}$? Explain.

b. What would be a good meaning of $27^{1/3}$? Of $64^{1/3}$?

c. Use a graphing calculator to find $8^{1/3}$, $27^{1/3}$, and $64^{1/3}$. Is the calculator interpreting $b^{1/3}$ as you would expect?

d. What would be a good meaning of $b^{1/3}$?

3. What would be a good meaning of $b^{1/n}$, where n is a counting number and b is nonnegative?

TIPS FOR SUCCESS: Make Changes

If you have not had passing scores on tests and quizzes during the first part of this course, it is time to determine what the problem is, what changes you should make, and whether you can commit to making those changes.

Sometimes students must change the way they study. For example, Rosie did poorly on exams and quizzes for the first third of the course. It was not clear why she was not passing, since she had good attendance, was actively involved in classroom work, and was doing the homework assignments. Suddenly Rosie started getting A's on every quiz and test. What had happened? Rosie said, "I figured out that, to do well, it was not enough practice to do just the exercises you assigned. Now I do a lot of extra exercises from each section."

HOMEWORK 4.1

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Simplify without using a calculator. Then use a calculator to verify your result. [**Note:** *To review order of operations, see Section A.6.*]

1. 2^{-1} **2.** 3^{-2} **3.** 5^0

4. $(-7)^0$ **5.** -4^2 **6.** -3^4

7. $(-4)^2$ **8.** $(-3)^4$ **9.** $\left(2^3\right)^2$

10. $\left(5^{-1}\right)^{-2}$ **11.** $2^{-1} + 3^{-1}$ **12.** $\dfrac{1}{2^{-1}} + \dfrac{1}{3^{-1}}$

Simplify without using a calculator.

13. $\dfrac{7^{902}}{7^{900}}$ **14.** $4^{2003}4^{-2000}$

15. $13^{500}13^{-500}$ **16.** $\left(130^{-1}\right)^{-1}$

17. $\left(25^3 - 411^5 + 89^2\right)^0$ **18.** $\dfrac{6^{200}}{2^{198}3^{199}}$

Simplify.

19. b^7b^{-9} **20.** b^4b^{-8}

21. $\left(7b^{-3}\right)\left(-2b^{-5}\right)$ **22.** $\left(4b^{-9}\right)\left(5b^4\right)$

23. $\left(-9b^{-7}c^5\right)\left(-8b^6c^{-5}\right)$ **24.** $\left(-4b^{-1}c^2\right)\left(6b^3c^{-4}\right)$

25. $\left(3b^2c^4\right)^3\left(2b^3c^5\right)^2$ **26.** $\left(4b^3c^7\right)^2\left(2b^5c^4\right)^3$

27. $3\left(b^5c\right)^{-2}$ **28.** $-6\left(bc^4\right)^{-3}$

29. $\left(2b^4c^{-2}\right)^5\left(3b^{-3}c^{-4}\right)^{-2}$ **30.** $\left(7b^{-4}c^{-1}\right)^{-2}\left(2b^3c^{-2}\right)^5$

31. $\dfrac{b^{-10}}{b^{15}}$ **32.** $\dfrac{b^{-2}}{b^2}$

33. $\dfrac{2b^{-12}}{5b^{-9}}$ **34.** $\dfrac{7b^{-4}}{6b^{-8}}$

35. $\dfrac{-12b^{-6}c^5}{14b^4c^5}$ **36.** $\dfrac{-28b^{-2}c^{-3}}{4b^{-3}c^{-1}}$

37. $\dfrac{15b^{-7}c^{-3}d^8}{-45c^2b^{-6}d^8}$ **38.** $\dfrac{18b^5c^3d^{-7}}{24b^{-6}c^3d^{-2}}$

39. $\dfrac{\left(-5b^{-3}c^4\right)\left(4b^{-5}c^{-1}\right)}{80b^2c^{17}}$ **40.** $\dfrac{\left(3b^4c^{-1}\right)\left(2b^{-7}c^{-8}\right)}{42b^{-5}c^4}$

41. $\dfrac{\left(24b^3c^{-6}\right)\left(49b^{-1}c^{-2}\right)}{\left(28b^2c^4\right)\left(14b^{-5}c\right)}$ **42.** $\dfrac{\left(16b^{-2}c\right)\left(25b^4c^{-5}\right)}{\left(15b^5c^{-1}\right)\left(8b^{-7}c^{-2}\right)}$

43. $\dfrac{\left(3b^5c^{-2}\right)^3}{2^{-1}b^{-3}c}$ **44.** $\dfrac{\left(2b^{-7}c^4\right)^4}{5^{-1}b^2c^6}$

45. $\dfrac{\left(2b^{-4}c\right)^{-3}}{\left(2b^2c^{-5}\right)^2}$ **46.** $\dfrac{\left(3bc^{-2}\right)^{-2}}{\left(3b^{-3}c\right)^{-1}}$

47. $\left(\dfrac{6b^5c^{-2}}{7b^2c^4}\right)^2$ **48.** $\left(\dfrac{2bc^{-7}}{5b^{-1}c^{-2}}\right)^3$

49. $\left(\dfrac{5b^4c^{-3}}{15b^{-2}c^{-1}}\right)^{-4}$ **50.** $\left(\dfrac{8b^{-2}c^2}{12b^{-5}c^{-3}}\right)^{-3}$

51. $\left(\dfrac{7b^4c^{-5}}{14b^7c^{-2}}\right)^0$ **52.** $\left(42b^{-8}c^7\right)^{-89}\left(42b^{-8}c^7\right)^{89}$

53. $b^{-1}c^{-1}$ **54.** $\dfrac{1}{b^{-1}} \cdot \dfrac{1}{c^{-1}}$

55. $\dfrac{1}{b^{-1}} + \dfrac{1}{c^{-1}}$ **56.** $b^{-1} + c^{-1}$

Simplify. Assume that n is a counting number.

57. $b^{4n}b^{3n}$ **58.** $b^{5n-1}b^{2n+4}$

59. $\dfrac{b^{7n-1}}{b^{2n+3}}$ **60.** $\dfrac{b^{3n+4}b^{n-5}}{b^{2n-3}}$

For $f(x) = 2(3)^x$ and $g(x) = 4^x$, find the following.

61. $f(3)$ **62.** $f(2)$ **63.** $f(-4)$ **64.** $f(-1)$

65. $g(a+2)$ **66.** $g(a+3)$ **67.** $g(2a)$ **68.** $g(3a)$

69. a. Complete Table 2 with output values of the function $f(x) = 2^x$. Then use a graphing calculator to verify your results.

Table 2 Input–Output Pairs of $f(x) = 2^x$ (Exercise 69)

x	$f(x)$	x	$f(x)$
−3		1	
−2		2	
−1		3	
0		4	

b. Plot the ordered pairs that you found in part (a). Then guess the graph of f and sketch it by hand. Use a graphing calculator to verify your graph.

c. Use your hand-drawn graph to estimate $2^{\frac{1}{2}}$.

70. a. Complete Table 3 with output values of the function $f(x) = \left(\dfrac{1}{2}\right)^x$. Then use a graphing calculator to verify your results.

Table 3 Input–Output Pairs of $f(x) = \left(\dfrac{1}{2}\right)^x$ (Exercise 70)

x	$f(x)$	x	$f(x)$
−4		0	
−3		1	
−2		2	
−1		3	

b. Plot the ordered pairs that you found in part (a). Then guess the graph of f and sketch it by hand. Use a graphing calculator to verify your graph.

c. Use your hand-drawn graph to estimate $\left(\dfrac{1}{2}\right)^{\frac{1}{2}}$.

Write the number in standard decimal form.

71. 3.965×10^2

72. 8.23172×10^3

73. 2.39×10^{-1}

74. 7.46×10^{-3}

75. 5.2×10^2

76. 7.74×10^6

77. 9.113×10^{-5}

78. 7.3558×10^{-2}

79. -6.52×10^{-4}

80. -3.006×10^{-3}

81. 9×10^5

82. 4×10^3

83. -8×10^0

84. -6.1×10^0

Write the number in scientific notation.

85. 54,260,000

86. 173,229

87. 23,587

88. 6,541,883

89. 0.00098

90. 0.08156

91. 0.0000346

92. 0.00000387

93. −42,215

94. −647,000

95. −0.00244

96. −0.000013

For Exercises 97 and 98, numbers are displayed in a graphing calculator table's version of scientific notation. Write each number in the Y_1 column in standard decimal form.

97. See Fig. 4.

98. See. Fig. 5.

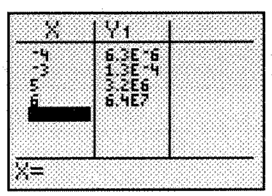

Figure 4 Exercise 97

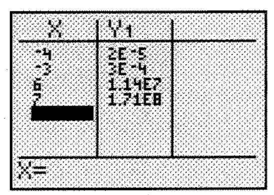

Figure 5 Exercise 98

For Exercises 99–102, the given sentence contains a number written in scientific notation. Write the number in standard decimal form.

99. The first evidence of life on Earth dates back to 3.6×10^9 years ago.

100. The Moon has an average distance from Earth of approximately 2.389×10^5 miles.

101. The hydrogen ion concentration in human blood is about 6.3×10^{-8} mole per liter.

102. The faintest sound that humans can hear has an intensity of about 10^{-12} watt per square meter.

For Exercises 103–106, the given sentence contains a number (other than a date) written in standard decimal form. Write the number in scientific notation.

103. The tanker *Exxon Valdez* spilled about 10,080,000 gallons of oil in Prince William Sound, Alaska, in 1989.

104. The average distance from Earth to Alpha Centauri is about 25,000,000,000,000 miles.

105. The wavelength of violet light is about 0.00000047 meter.

106. One second is about 0.0000000317 year.

107. The numbers of bald eagle pairs in the continental United States are shown in Table 4 for various years.

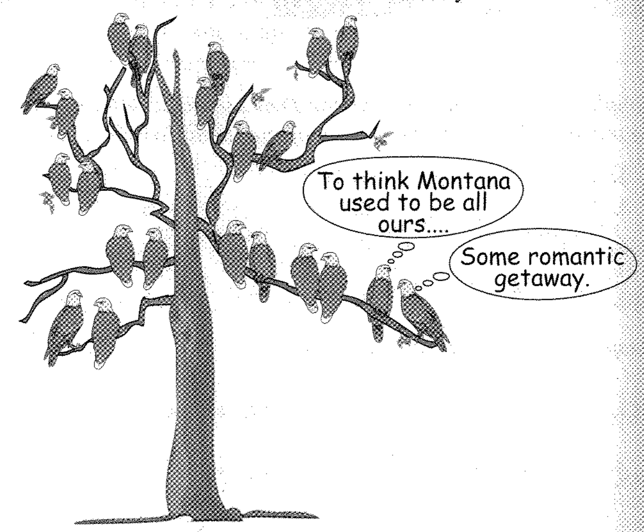

Table 4 Numbers of Bald Eagle Pairs

Year	Number of Bald Eagle Pairs (thousands)
1963	0.4
1974	0.8
1981	1.2
1986	1.9
1990	3.0
1995	4.7
2000	6.5
2005	7.7

Source: *U.S. Fish and Wildlife Service*

Let n be the number of bald eagle pairs (in thousands) in the continental United States at t years since 1960. The situation can be described by the linear function $n = 0.18t - 1.63$ and the exponential function $n = 0.29(1.078)^t$.

a. Use a graphing calculator to draw the graphs of the two functions and, in the same viewing window, the scattergram of the data. Which function describes the situation better?

b. Use the exponential model to predict the number of bald eagle pairs in 2012.

c. Use the linear model to predict the number of bald eagle pairs in 2012. Explain why your result is so much smaller than your result from part (b). [**Hint:** Zoom Out at least once.]

108. The sales of digital TV sets and displays are shown in Table 5 for various years.

Table 5 Sales of Digital TV Sets and Displays

Year	Number of Digital TV Sets and Displays Sold (millions)
2000	0.6
2001	1.4
2002	2.5
2003	4.0
2004	7.5
2005	15.0

Source: *Consumer Electronics Association*

Let n be the sales of digital TVs and displays (in millions) in the year that is t years since 2000. The situation can be described by the linear function $n = 2.62t - 1.39$ and the exponential function $n = 0.67(1.85)^t$.

a. Use a graphing calculator to draw the graphs of the two functions and, in the same viewing window, the scattergram of the data. Which function describes the situation better?

b. Use the exponential model to predict the sales of digital TVs and displays in 2011.

c. Use the linear model to predict the sales of digital TVs and displays in 2011. Explain why your result is so much smaller than your result from part (b). [**Hint:** Zoom Out at least once.]

109. Two students try to simplify $(5b^2)^{-1}$:

Student A

$$(5b^2)^{-1} = -5b^{-2} = \frac{-5}{b^2}$$

Student B

$$(5b^2)^{-1} = 5^{-1}(b^2)^{-1} = 5^{-1}b^{-2} = \frac{1}{5b^2}$$

Did either student simplify the expression correctly? Describe any errors.

110. Two students try to simplify an expression:

Student 1

$$\frac{7b^8}{b^{-3}} = 7b^{8-(-3)} = 7b^{11}$$

Student 2

$$\frac{7b^8}{b^{-3}} = 7b^{8-3} = 7b^5$$

Did either student simplify the expression correctly? Describe any errors.

111. A student tries to simplify $\dfrac{3b^{-2}c^4}{d^7}$:

$$\frac{3b^{-2}c^4}{d^7} = \frac{c^4}{3b^2d^7}$$

Describe any errors. Then simplify the expression correctly.

112. A student tries to simplify $(7x^4)^5$:

$$(7x^4)^5 = 7(x^4)^5 = 7x^{20}$$

Describe any errors. Then simplify the expression correctly.

113. It is common to confuse expressions such as $2^2, 2^{-1}, 2(-1), \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^{-1}, -2^2, (-2)^2$, and $\frac{1}{2}$. List these numbers from least to greatest. Are there any "ties"?

114. a. Simplify $\left(\dfrac{b}{c}\right)^{-2}$.

b. Simplify $\left(\dfrac{b}{c}\right)^{-n}$.

c. Use your result from part (b) to simplify $\left(\dfrac{b}{c}\right)^{-5}$ in one step.

115. Explore "0^0":

a. Simplify 5^0, 4^0, 3^0, 2^0, and 1^0. On the basis of these values, what would be a reasonable value of 0^0?

b. Simplify 0^5, 0^4, 0^3, 0^2, and 0^1. On the basis of these values, what would be a reasonable value of 0^0?

c. Why is it a good idea to leave 0^0 meaningless?

116. Simplify each expression.

a. b^{-1}

b. $(b^{-1})^{-1}$

c. $((b^{-1})^{-1})^{-1}$

d. $(((b^{-1})^{-1})^{-1})^{-1}$

e. $\underbrace{((((b^{-1})^{-1})^{-1})\cdots)^{-1}}_{n\text{ exponents}}$

117. It is a common error to confuse the properties $b^m b^n = b^{m+n}$ and $(b^m)^n = b^{mn}$. Explain why each property makes sense, and compare the properties. Give examples to illustrate your comparison. (See page 4 for guidelines on writing a good response.)

118. Describe what it means to use exponential properties to simplify an expression. Include several examples in your description. (See page 4 for guidelines on writing a good response.)

Related Review

For $f(x) = 2x$ and $g(x) = 2^x$, find the following.

119. $f(3)$

120. $f(-3)$

121. $g(3)$

122. $g(-3)$

Expressions, Equations, Functions, and Graphs

Perform the indicated instruction. Then use words such as linear, exponential, function, one variable, *and* two variables *to describe the expression, equation, or system. For instance, to describe $2x = 10$, you could say "$2x = 10$ is a linear equation in one variable."*

123. Solve:

$$y = 3x + 1$$
$$y = 2x - 4$$

124. Simplify $5(3x + 1) - 4(2x - 4)$.

125. Solve $3x + 1 = 2x - 4$.

126. Graph $f(x) = 3x + 1$ by hand.

4.2 RATIONAL EXPONENTS

Objectives

- Know definitions of *rational exponents.*
- Simplify expressions that have rational exponents.

In Section 4.1, we worked with integer exponents. In this section, we work with exponents that are rational numbers (Section A.2).

Definitions of Rational Exponents

How should we define $b^{1/n}$, where n is a counting number? If the exponential property $(b^m)^n = b^{mn}$ is to be true for $m = \frac{1}{2}$ and $n = 2$, then

$$\left(9^{\frac{1}{2}}\right)^2 = 9^{\frac{1}{2}\cdot 2} = 9^1 = 9$$

Since $(-3)^2 = 9$ and $3^2 = 9$, the statement suggests that a good meaning of $9^{1/2}$ is -3 or 3. We define $9^{1/2} = 3$. We call the nonnegative number 3 the *principal second root,* or **principal square root,** of 9, written $\sqrt{9}$.

Similarly, if the property $(b^m)^n = b^{mn}$ is to be true for $m = \frac{1}{3}$ and $n = 3$, then

$$\left(8^{\frac{1}{3}}\right)^3 = 8^{\frac{1}{3}\cdot 3} = 8^1 = 8$$

Since $2^3 = 8$, the statement suggests that a good meaning of $8^{1/3}$ is 2. The number 2 is called the *third root,* or **cube root,** of 8, written $\sqrt[3]{8}$.

For $(-8)^{1/3}$, a good meaning is -2, since $(-2)^3 = -8$. We do not assign a real-number value to $(-9)^{1/2}$, since no real number squared is equal to -9.

DEFINITION $b^{1/n}$

For the counting number n, where $n \neq 1$,

- If n is odd, then $\boldsymbol{b^{1/n}}$ is the number whose nth power is b, and we call $b^{1/n}$ the **nth root of b.**
- If n is even and $b \geq 0$, then $\boldsymbol{b^{1/n}}$ is the nonnegative number whose nth power is b, and we call $b^{1/n}$ the **principal nth root of b.**
- If n is even and $b < 0$, then $b^{1/n}$ is not a real number.

$\boldsymbol{b^{1/n}}$ may be represented by $\sqrt[n]{b}$.

Example 1 Simplifying Expressions Involving Rational Exponents

Simplify.

1. $25^{1/2}$ **2.** $64^{1/3}$ **3.** $(-64)^{1/3}$
4. $16^{1/4}$ **5.** $-16^{1/4}$ **6.** $(-16)^{1/4}$

Solution

1. $25^{1/2} = 5$, since $5^2 = 25$.
2. $64^{1/3} = 4$, since $4^3 = 64$.
3. $(-64)^{1/3} = -4$, since $(-4)^3 = -64$.
4. $16^{1/4} = 2$, since $2^4 = 16$.
5. $-16^{1/4} = -(16^{1/4}) = -2$.
6. $(-16)^{1/4}$ is not a real number, since the fourth power of any real number is nonnegative.

Graphing calculator checks for Problems 1, 2, and 3 are shown in Fig. 6. For example, to find $25^{1/2}$, press **25** [∧] [(] **1** [÷] **2** [)] [ENTER]. If an exponent involves an operation, you must use parentheses. ■

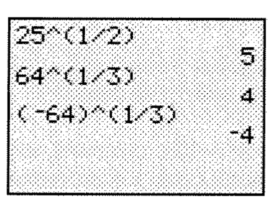

Figure 6 Checks for Problems 1, 2, and 3

What would be a reasonable definition of $b^{m/n}$? If the properties of exponents we discussed in Section 4.1 are to hold true for rational exponents, we have

$$8^{\frac{2}{3}} = 8^{\frac{1}{3}\cdot 2} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4 \quad \text{or} \quad 8^{\frac{2}{3}} = 8^{2\cdot\frac{1}{3}} = \left(8^2\right)^{\frac{1}{3}} = 64^{\frac{1}{3}} = 4$$

Likewise,

$$32^{\frac{3}{5}} = 32^{\frac{1}{5}\cdot 3} = \left(32^{\frac{1}{5}}\right)^3 = 2^3 = 8 \quad \text{or} \quad 32^{\frac{3}{5}} = 32^{3\cdot\frac{1}{5}} = \left(32^3\right)^{\frac{1}{5}} = 32{,}768^{\frac{1}{5}} = 8$$

Also,

$$32^{-\frac{3}{5}} = \frac{1}{32^{\frac{3}{5}}} = \frac{1}{8}$$

DEFINITION Rational exponent

For the counting numbers m and n, where $n \neq 1$ and b is any real number for which $b^{1/n}$ is a real number,

- $\boldsymbol{b^{m/n} = (b^{1/n})^m = (b^m)^{1/n}}$
- $\boldsymbol{b^{-m/n} = \frac{1}{b^{m/n}}}$, $b \neq 0$

A power of the form $b^{m/n}$ or $b^{-m/n}$ is said to have a **rational exponent.**

Example 2 Simplifying Expressions Involving Rational Exponents

Simplify.

1. $25^{3/2}$ **2.** $(-27)^{2/3}$ **3.** $32^{-2/5}$ **4.** $(-8)^{-5/3}$

Solution

1. $25^{3/2} = \left(25^{1/2}\right)^3 = 5^3 = 125$
2. $(-27)^{2/3} = \left((-27)^{1/3}\right)^2 = (-3)^2 = 9$
3. $32^{-2/5} = \dfrac{1}{32^{2/5}} = \dfrac{1}{\left(32^{1/5}\right)^2} = \dfrac{1}{2^2} = \dfrac{1}{4}$
4. $(-8)^{-5/3} = \dfrac{1}{(-8)^{5/3}} = \dfrac{1}{\left((-8)^{1/3}\right)^5} = \dfrac{1}{(-2)^5} = \dfrac{1}{-32} = -\dfrac{1}{32}$

Graphing calculator checks for Problems 1, 2, and 3 are shown in Fig. 7. ■

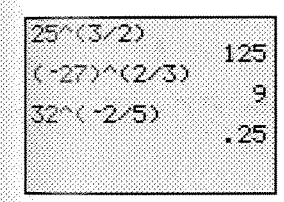

Figure 7 Checks for Problems 1, 2, and 3

Example 3 Evaluating an Exponential Function

For $f(x) = 64^x$, $g(x) = 3(16)^x$, and $h(x) = -5(9)^x$, find the following.

1. $f\left(\frac{2}{3}\right)$ **2.** $g\left(\frac{3}{4}\right)$ **3.** $h\left(-\frac{1}{2}\right)$

Solution

1. $f\left(\frac{2}{3}\right) = 64^{2/3} = \left(64^{1/3}\right)^2 = 4^2 = 16$
2. $g\left(\frac{3}{4}\right) = 3(16)^{3/4} = 3\left(16^{1/4}\right)^3 = 3(2)^3 = 3 \cdot 8 = 24$
3. $h\left(-\frac{1}{2}\right) = -5(9)^{-1/2} = \dfrac{-5}{9^{1/2}} = -\dfrac{5}{3}$ ■

Properties of Rational Exponents

The properties of exponents that we discussed in Section 4.1 are valid for *rational* exponents.

Properties of Rational Exponents

If m and n are rational numbers and b and c are any real numbers for which b^m, b^n, and c^n are real numbers, then

- $b^m b^n = b^{m+n}$ Product property for exponents
- $\dfrac{b^m}{b^n} = b^{m-n}, b \neq 0$ Quotient property for exponents
- $(bc)^n = b^n c^n$ Raising a product to a power
- $\left(\dfrac{b}{c}\right)^n = \dfrac{b^n}{c^n}, \quad c \neq 0$ Raising a quotient to a power
- $(b^m)^n = b^{mn}$ Raising a power to a power

We can use properties of exponents to help us simplify expressions involving rational exponents.

Example 4 Simplifying Expressions Involving Rational Exponents

Simplify. Assume that b is positive.

1. $(4b^6)^{3/2}$ **2.** $\dfrac{b^{2/7}}{b^{-3/7}}$

Solution

1.
$$\begin{aligned} (4b^6)^{3/2} &= 4^{3/2}(b^6)^{3/2} && \text{Raise factors to } n\text{th power: } (bc)^n = b^n c^n \\ &= (4^{1/2})^3 b^{\frac{6}{1}\cdot\frac{3}{2}} && b^{m/n} = (b^{1/n})^m\text{; multiply exponents: } (b^m)^n = b^{mn} \\ &= 2^3 b^9 && 4^{\frac{1}{2}} = 2\text{; multiply.} \\ &= 8b^9 && \text{Simplify.} \end{aligned}$$

2.
$$\begin{aligned} \frac{b^{2/7}}{b^{-3/7}} &= b^{\frac{2}{7}-\left(-\frac{3}{7}\right)} && \text{Subtract exponents: } \frac{b^m}{b^n} = b^{m-n} \\ &= b^{\frac{2}{7}+\frac{3}{7}} && \text{Simplify.} \\ &= b^{5/7} && \text{Add.} \end{aligned}$$

■

Example 5 Simplifying Expressions Involving Rational Exponents

Simplify. Assume that b is positive.

1. $b^{2/3}b^{1/2}$ **2.** $\left(\dfrac{32b^2}{b^{12}}\right)^{2/5}$

Solution

1.
$$\begin{aligned} b^{2/3}b^{1/2} &= b^{\frac{2}{3}+\frac{1}{2}} && \text{Add exponents: } b^m b^n = b^{m+n} \\ &= b^{\frac{4}{6}+\frac{3}{6}} && \text{Find common denominator.} \\ &= b^{7/6} && \text{Add.} \end{aligned}$$

2.
$$\begin{aligned} \left(\frac{32b^2}{b^{12}}\right)^{2/5} &= (32b^{2-12})^{2/5} && \text{Subtract exponents: } \frac{b^m}{b^n} = b^{m-n} \\ &= (32b^{-10})^{2/5} && \text{Subtract.} \\ &= \left(\frac{32}{b^{10}}\right)^{2/5} && \text{Write powers so exponents are positive: } b^{-n} = \frac{1}{b^n} \\ &= \frac{32^{2/5}}{(b^{10})^{2/5}} && \text{Raise numerator and denominator to } n\text{th power: } \left(\frac{b}{c}\right)^n = \frac{b^n}{c^n} \\ &= \frac{(32^{1/5})^2}{b^{10\cdot\frac{2}{5}}} && b^{m/n} = (b^{1/n})^m\text{; multiply exponents: } (b^m)^n = b^{mn} \\ &= \frac{2^2}{b^4} && 32^{1/5} = 2\text{; multiply.} \\ &= \frac{4}{b^4} && \text{Simplify.} \end{aligned}$$

■

Example 6 Simplifying an Expression Involving Rational Exponents

Simplify $\dfrac{(81b^6c^{20})^{1/2}}{(27b^{12}c^9)^{2/3}}$. Assume that b and c are positive.

Solution

$$\begin{aligned} \frac{(81b^6c^{20})^{1/2}}{(27b^{12}c^9)^{2/3}} &= \frac{81^{1/2}(b^6)^{1/2}(c^{20})^{1/2}}{27^{2/3}(b^{12})^{2/3}(c^9)^{2/3}} && \text{Raise factors to a power: } (bc)^n = b^n c^n \\ &= \frac{9b^{6\cdot\frac{1}{2}}c^{20\cdot\frac{1}{2}}}{(27^{1/3})^2 b^{12\cdot\frac{2}{3}}c^{9\cdot\frac{2}{3}}} && 81^{1/2} = 9;\ b^{m/n} = (b^{1/n})^m\text{; multiply exponents: } (b^m)^n = b^{mn} \\ &= \frac{9b^3c^{10}}{3^2b^8c^6} && 27^{1/3} = 3\text{; multiply.} \\ &= \frac{9b^{-5}c^4}{9} && \text{Subtract exponents: } \frac{b^m}{b^n} = b^{m-n} \\ &= \frac{c^4}{b^5} && \text{Write powers so exponents are positive: } b^{-n} = \frac{1}{b^n} \end{aligned}$$

■

group exploration

Looking ahead: Graphical significance of a and b for $y = ab^x$

1. Use ZDecimal to graph these equations of the form $y = b^x$ in order, and describe what you observe:

$$y = 1.2^x, \quad y = 1.5^x, \quad y = 2^x, \quad \text{and} \quad y = 5^x$$

If you want a better view, set Ymin = 0. To change window settings, see Section B.7.

Do the same with the equations

$$y = 0.3^x, \quad y = 0.5^x, \quad y = 0.7^x, \quad \text{and} \quad y = 0.9^x$$

2. Use ZStandard to graph these equations of the form $y = a(1.1)^x$ in order, and describe what you observe:

$$y = 2(1.1)^x, \quad y = 3(1.1)^x, \quad y = 4(1.1)^x, \quad \text{and} \quad y = 5(1.1)^x$$

If you want a better view, set Ymin = 0.
Use ZStandard to do the same with the equations

$$y = -2(1.1)^x, \quad y = -3(1.1)^x, \quad y = -4(1.1)^x, \quad \text{and} \quad y = -5(1.1)^x$$

If you want a better view, set Ymax = 0.

3. So far, you have sketched the graphs of equations of only the forms $y = b^x$ (where $a = 1$) and $y = a(1.1)^x$ (where $b = 1.1$). Graph more equations of the form $y = ab^x$, until you are confident that you know the graphical significance of the constants a and b, for any possible combination of values of a and b. If you have any new insights into the graphical significance of a and b, describe those insights.

4. Describe the graph of $y = ab^x$ in the following situations.
 - **a.** a is positive
 - **b.** a is negative
 - **c.** $b > 1$
 - **d.** $0 < b < 1$
 - **e.** $b = 1$
 - **f.** b is negative

5. Describe the connection between the y-intercept of $y = ab^x$ and the values of a and b.

group exploration

Looking ahead: Numerical significance of a and b for $f(x) = ab^x$

In this exploration, you will investigate the nature of exponential functions of the form $f(x) = ab^x$.

1. Use a graphing calculator to create a table of ordered pairs for $f(x) = 2(3)^x$, $g(x) = 64\left(\frac{1}{2}\right)^x$, and a third exponential function of your choice. (See Figs. 8 and 9.) Use the following values for the x-coordinates: 0, 1, 2, . . . , 6.

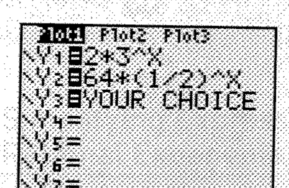

Figure 8 Enter the three functions

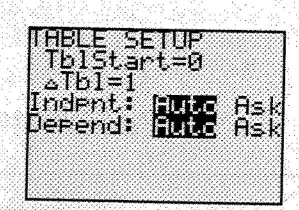

Figure 9 Table setup

2. **a.** What connection do you notice between the y-coordinates of each function and the base b of the function $y = ab^x$?
 b. Test the connection you described in part (a) by choosing yet another exponential function, and check whether it behaves as you think it should.
 c. For $f(x) = ab^x$, we have $f(0) = a$, $f(1) = ab$, $f(2) = abb$, and $f(3) = abbb$. Explain why these results suggest that your response to part (a) is correct.

3. **a.** What connection do you notice between the y-coordinates of each function and the coefficient a of the function $y = ab^x$?
 b. Test the connection you described in part (a) by choosing yet another exponential function, and check whether it behaves as you think it should.
 c. Use pencil and paper to find $f(0)$, where $f(x) = ab^x$. Explain why your result shows that your response to part (a) is correct.

TIPS FOR SUCCESS: Complete Exercises Without Help

If you work an exercise by referring to a similar example in your notebook or in the text, try the exercise again without that help. If you need to refer to your source of help to solve the exercise a second time, try the exercise a third time without help. When you complete the exercise without help, reflect on which concepts you used to work the exercise, where you had difficulty, and what key idea opened the door of understanding for you. You can use a similar strategy in getting help from another student, an instructor, or a tutor.

If this sounds like a lot of work, it is! But this work is well worth it. Although it is important to complete each assignment, it is also important to learn as much as possible while progressing through it.

HOMEWORK 4.2

FOR EXTRA HELP ▸ Student Solutions Manual · PH Math/Tutor Center · MathXL® · MyMathLab

*Simplify without using a calculator. Then use a graphing calculator to verify your result. [**Graphing Calculator:** Instructions for $x^{m/n}$: Press* [X, T, Θ, n] [^] [(] ***m*** [÷] ***n*** [)]*.]*

1. $16^{1/2}$	**2.** $27^{1/3}$	**3.** $1000^{1/3}$
4. $32^{1/5}$	**5.** $49^{1/2}$	**6.** $81^{1/4}$
7. $125^{1/3}$	**8.** $64^{1/6}$	**9.** $8^{4/3}$
10. $16^{3/4}$	**11.** $9^{3/2}$	**12.** $64^{2/3}$
13. $32^{2/5}$	**14.** $27^{4/3}$	**15.** $4^{5/2}$
16. $81^{3/4}$	**17.** $27^{-1/3}$	**18.** $16^{-1/4}$
19. $-36^{-1/2}$	**20.** $-32^{-1/5}$	**21.** $4^{-5/2}$
22. $9^{-3/2}$	**23.** $(-27)^{-4/3}$	**24.** $(-32)^{-3/5}$

Simplify without using a calculator. Then use a graphing calculator to verify your result.

25. $2^{1/4}2^{3/4}$	**26.** $3^{7/5}3^{3/5}$	**27.** $\left(3^{1/2}2^{3/2}\right)^2$
28. $\left(2^{2/3}5^{1/3}\right)^3$	**29.** $\dfrac{7^{1/3}}{7^{-5/3}}$	**30.** $\dfrac{5^{4/3}}{5^{1/3}}$

For $f(x) = 81^x$, $g(x) = 4(27)^x$, and $h(x) = -2(4)^x$, find the following.

31. $f\left(\frac{3}{4}\right)$	**32.** $f\left(\frac{1}{4}\right)$	**33.** $g\left(\frac{1}{3}\right)$
34. $g\left(\frac{2}{3}\right)$	**35.** $g\left(-\frac{1}{3}\right)$	**36.** $g\left(-\frac{2}{3}\right)$
37. $h\left(\frac{3}{2}\right)$	**38.** $h\left(\frac{5}{2}\right)$	

39. Without using a calculator, complete Table 6 with values of the function $f(x) = 16^x$. Then use a graphing calculator to verify your results.

Table 6 Values of the Function $f(x) = 16^x$

x	$f(x)$	x	$f(x)$
$-\frac{3}{4}$		$\frac{1}{4}$	
$-\frac{1}{2}$		$\frac{1}{2}$	
$-\frac{1}{4}$		$\frac{3}{4}$	
0		1	

40. Without using a calculator, complete Table 7 with values of the function $f(x) = 64^x$. Then use a graphing calculator to verify your results.

Table 7 Values of the Function $f(x) = 64^x$

x	$f(x)$	x	$f(x)$
$-\frac{5}{6}$		$\frac{1}{6}$	
$-\frac{2}{3}$		$\frac{1}{3}$	
$-\frac{1}{2}$		$\frac{1}{2}$	
$-\frac{1}{3}$		$\frac{2}{3}$	
$-\frac{1}{6}$		$\frac{5}{6}$	
0		1	

Simplify. Assume that b and c are positive.

41. $b^{7/6}b^{5/6}$

42. $b^{1/5}b^{3/5}$

43. $b^{3/5}b^{-13/5}$

44. $b^{2/7}b^{-6/7}$

45. $(16b^8)^{1/4}$

46. $(27b^{27})^{1/3}$

47. $4(25b^8c^{14})^{-1/2}$

48. $-(8b^{-6}c^{12})^{2/3}$

49. $(b^{3/5}c^{-1/4})(b^{2/5}c^{-7/4})$

50. $(b^{-4/3}c^{1/2})(b^{-2/3}c^{-3/2})$

51. $(5bcd)^{1/5}(5bcd)^{4/5}$

52. $(6bc^2)^{5/7}(6bc^2)^{2/7}$

53. $[(3b^5)^3(3b^9c^8)]^{1/4}$

54. $[(4b^3)^2(b^2c^{12})]^{1/4}$

55. $\dfrac{b^{-2/5}c^{11/8}}{b^{18/5}c^{-5/8}}$

56. $\dfrac{b^{3/4}c^{1/2}}{b^{-1/4}c^{-1/2}}$

57. $\left(\dfrac{9b^3c^{-2}}{25b^{-5}c^4}\right)^{-1/2}$

58. $\left(\dfrac{16b^{12}c^2}{2b^{-3}c^{-4}}\right)^{-1/3}$

59. $32^{1/5}b^{3/7}b^{2/5}$

60. $16^{1/4}b^{1/4}b^{1/3}$

61. $\dfrac{b^{5/6}}{b^{1/4}}$

62. $\dfrac{b^{-2/3}}{b^{1/7}}$

63. $\dfrac{(9b^5)^{3/2}}{(27b^4)^{2/3}}$

64. $\dfrac{(32b^3)^{3/5}}{(16b^3)^{3/2}}$

65. $\left(\dfrac{8b^{2/3}}{2b^{4/5}}\right)^{3/2}$

66. $\left(\dfrac{27b^{1/3}c^{3/4}}{8b^{-2/3}c^{1/2}}\right)^{4/3}$

67. $\dfrac{(8bc^3)^{1/3}}{(81b^{-5}c^3)^{3/4}}$

68. $\dfrac{(1000b^{-7}c^8)^{2/3}}{(32b^{15}c^4)^{3/5}}$

69. $b^{2/5}(b^{8/5} + b^{3/5})$

70. $c^{1/3}(c^{8/3} - c^{5/3})$

71. The numbers of countries that have participated in the Winter Olympics are shown in Table 8 for various years.

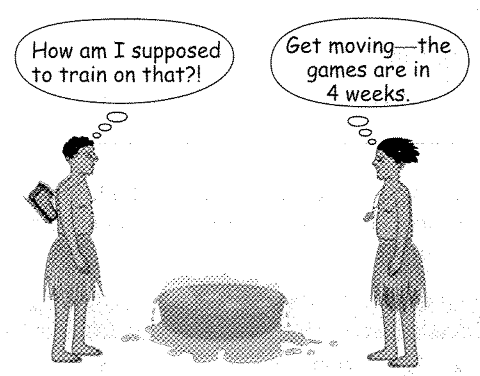

Table 8 Numbers of Countries Participating in Winter Olympics

Year	Number of Countries
1924	16
1948	28
1968	37
1988	57
2006	85

Source: The Complete Book of the Winter Olympics

Let n be the number of countries participating in the Winter Olympics at t years since 1900. The situation can be described by the linear function $n = 0.81t - 9.41$ and the exponential function $n = 10.1(1.02)^t$.

a. Use a graphing calculator to draw the graphs of the two functions and, in the same viewing window, the scattergram of the data. Which function describes the situation better?

b. Use the exponential model to predict the number of countries that will participate in the 2010 Winter Olympics.

c. Use the exponential model and "intersect" on a graphing calculator to estimate when 42 countries participated in the Winter Olympics. [**Hint:** Graph the model and the horizontal line $n = 42$.]

72. The numbers of high school students who take college-level courses and test for credit are shown in Table 9 for various years.

Table 9 Numbers of High School Students Taking College-Level Courses and Testing for Credit

Year	Number of Students Who Test for Credit (thousands)
1963	21.8
1970	55.4
1977	82.7
1984	177.4
1991	359.1
1998	635.2
2005	1200.0

Source: *The College Board*

Let n be the number (in thousands) of high school students who take college-level courses and test for credit in the year that is t years since 1960. The situation can be described by the linear function $n = 25.36t - 246.99$ and the exponential function $n = 18.36(1.098)^t$.

a. Use a graphing calculator to draw the graphs of the two functions and, in the same viewing window, the scattergram of the data. Which function describes the situation better?

b. Use the exponential model to predict the number of students who will take college-level courses and test for credit in 2011.

c. Use the exponential model and "intersect" on a graphing calculator to estimate in which year 1000 high school students took college-level courses and tested for credit. [**Hint:** Graph the model and the horizontal line $n = 1000$.]

73. We can represent $\sqrt{5}$ by $5^{1/2}$. Explain.

74. To use a graphing calculator to find that $16^{1/2} = 4$, we press 16 [∧] [(] 1 [÷] 2 [)]. If we omit the parentheses, we get the incorrect result 8. Explain why.

75. To convert from the scientific notation $N \times 10^k$ to standard decimal notation, we move the decimal point of the number N to the right by k places if k is positive. Explain.

76. List the exponent definitions and properties that are discussed in this section and Section 4.1. Explain how you can recognize which definition or property will help you simplify a given expression.

Related Review

For $f(x) = 8x$ *and* $g(x) = 8^x$, *find the following.*

77. $f\left(\frac{1}{3}\right)$

78. $f\left(\frac{4}{3}\right)$

79. $g\left(\frac{1}{3}\right)$

80. $g\left(\frac{4}{3}\right)$

81. $f\left(-\frac{1}{3}\right)$

82. $f\left(-\frac{2}{3}\right)$

83. $g\left(-\frac{1}{3}\right)$

84. $g\left(-\frac{2}{3}\right)$

Expressions, Equations, Functions, and Graphs

Perform the indicated instruction. Then use words such as linear, exponential, function, one variable, *and* two variables *to describe the expression, equation, or system. For instance, to describe* $f(x) = 7(4)^x$, *you could say "*$f(x) = 7(4)^x$ *is an exponential function."*

85. Graph $f(x) = \frac{3}{2}x - 4$ by hand.

86. Solve:

$$y = \frac{3}{2}x - 4$$
$$y = -\frac{1}{4}x + 3$$

87. Let $f(x) = \frac{3}{2}x - 4$. Find x when $f(x) = 5$.

88. Solve $\frac{3}{2}x - 4 = -\frac{1}{4}x + 3$.

4.3 GRAPHING EXPONENTIAL FUNCTIONS

Objectives

- Sketch the graph of an exponential function.
- Know the graphical significance of a and b for a function of the form $f(x) = ab^x$.
- Know the *base multiplier property*, the *increasing or decreasing property*, and the *reflection property*.

Recall from Section 4.1 that an exponential function is a function whose equation can be put into the form $f(x) = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$. In this section, we discuss how to use the values of a and b to help us graph an exponential function.

Graphing Exponential Functions

When graphing a certain type of function for the first time, we often begin by finding outputs for integer inputs near zero.

Example 1 Graphing an Exponential Function with $b > 1$

Graph $f(x) = 2^x$ by hand.

Solution

First, we list input–output pairs of the function f in Table 10. Note that as the value of x increases by 1, the value of y is multiplied by 2 (the base).

Next, we plot the solutions from Table 10 in Fig. 10 and sketch an increasing curve that contains the plotted points. The graph shows that as the value of x increases by 1, the value of y is doubled.

Table 10 Input–Output Pairs of $f(x) = 2^x$

x	$f(x)$
−3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
−2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
−1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

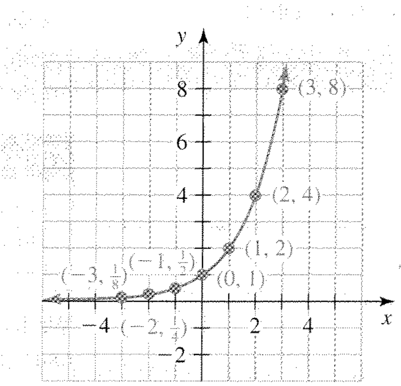

Figure 10 Graph of $f(x) = 2^x$